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ABSTRACT

This is the first unit of a 15-unit School Mathematics Study Group (SMSG) mathematics text for high school students. Topics presented in the first chapter (Structuring Space) include: lines and points; planes; intersections; intersections of lines and planes; betweenness and segments; separation; angles; locating positions and points; coordinates; coordinates in the plane; and graphs in the plane. Topics presented in the second chapter (Functions) include car travel; falling objects; examples of functions; ways of representing functions; discovering the usefulness of graphs; the identify functions; and new kinds of functions. (JN)

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UNIT ONE

Chapter 1. Structuring Space

Chapter 2. Functions

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Chapter 1

STRUCTURING SPACE

1-1. Introduction

Geometry began a few thousand years ago when man began to study systematically the size, shape, and location of objects in his physical surroundings. This understanding was necessary then as it is today in such activities as navigation, building, and surveying. Today, however, we find that geometry has been considerably refined and extended to deal with the more complicated problems which confront us.

Geometry is a system of ideas based on our experiences with physical objects. For example, from our experiences with various boxes we have developed an idea of the shape and form of a box. This idea we have of a box is called a geometric figure and it is this figure, not the box which is studied in geometry. In other words, geometry is developed as a mathematical model of our experience with physical space.

The idea of a point in geometry is formed from such objects as the end of a pin, a speck of dust, a grain of sand, and a distant star. Consequently, we may think of a point as a definite location in space.

Points are represented by dots and are named by capital letters as is shown for point A at the left.

A •

We think of space as being a set of points. There are an unlimited number of points in space. Suppose that a mosquito is flying about inside a room. Its location at a particular instant might be described as a point and all such points make up a set. All the points inside of this room make up a portion or subset of space and all the points outside this room make up another portion or subset of space. Therefore in geometry, space is the set of all points.

Each of the various kinds of geometric figures studied here will be developed as a set of points. We shall see how space is structured by carefully examining the way in which the various figures are formed and related to each other using the basic ideas of point, line, and plane. Later in this

chapter a systematic way of assigning numbers to points in a plane will be developed. This will provide us with another very useful method of studying the structure of space.

Exercises 1-1

(Class Discussion)

1. Complete the following sentence.
   Geometry is the study of _____?_____.

2. What is the relationship between geometry and physical objects?

3. Describe three objects which can be used to form the idea of a point, other than the objects already mentioned.

4. What meaning would a point have for a surveyor or navigator? Give some examples.

5. If you were to think of a point as a definite location, how would this affect the way in which you would represent a point?

6. In forming the idea of a box as a geometric figure, what are some things about a box you would ignore?

1-2. Lines and Points

The points shown at the left lie on what we think of as a "straight line", or simply a "line".

There are an unlimited number of points on a line, and a line extends without ending in both directions.

Think of two boys holding a string stretched between them as shown below.

When the string is stretched tight between the boys' hands, the stretched string represents part of a line. The line itself, however, goes beyond the boys' hands without ending.

Some other objects which help to form the idea of a line are

(1) the edge of a ruler,

(2) the crease of a carefully folded paper,

(3) the edge of a box,

(4) a flagpole.

The drawing below representing line ℓ can be made by tracing a pencil along a straightedge on a sheet of paper.

What do the arrowheads on the drawing indicate?

Mark a point P on your paper and draw three lines, ℓ, m, and n which contain point P. Are there other lines on which point P lies?

We can think of many lines passing through a point. For two different points, however, we think of only one line passing through them since we think of a line as being straight. We shall accept the following statement as a fundamental characteristic or property of the structure of our space.

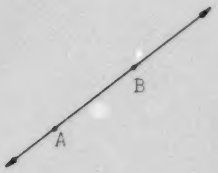
Property 1: Through any two different points in space there is exactly one line.

This property may be restated in the following ways.

(1) For any two points there is one line and only one line containing them.

(2) Any two points determine a line.

As a result of Property 1, a line can be clearly identified by naming any two of its points.



For example, the line shown at the left can be identified as "line AB" which is written "$\overleftrightarrow{AB}$".

It may also be identified as "line BA" and written "$\overleftrightarrow{BA}$".

Exercises 1-2a

(Class Discussion)

1. The following drawing was made on the blackboard with a broad piece of chalk. It represents two lines ℓ and m that cross or intersect.

(a) How does this drawing suggest the possibility that the intersection of ℓ and m may contain several points?

(b) Let's explore this possibility by assuming that the intersection of ℓ and m contains two points, A and B.

Show that this means we have two lines passing through the same two points.

But Property 1 states that there can be only one line passing through any two points!

(c) If we wish to continue to include Property 1 in the structure of our space why must we exclude the possibility of there being two points in the intersection of two lines?

What about the possibility of there being more than two points in the intersection?

We have the following statement as a rather surprising result of Property 1.

If two lines intersect, they intersect in exactly one point which is called the point of intersection.

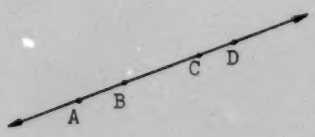
This statement together with Property 1 can be summarized as follows.

Two points determine a line and two intersecting lines determine a point.

Exercises 1-2b

1. Name three objects other than those already mentioned which suggest the idea of a line.

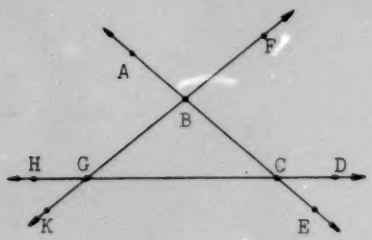
2. Points A, B, C, and D lie on the same line.



(a) Using these points, write five different names for this line.

(b) How many different names are there using these points?

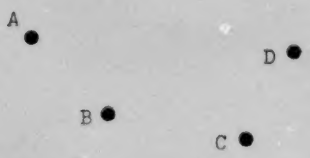
3. (a) How many lines are shown below?



(b) Name the lines.

(c) Name the point of intersection for each pair of lines.

4. Copy the set of points {A,B,C,D} arranged as shown.



(a) Draw <u>all</u> the lines you can through pairs of these points.

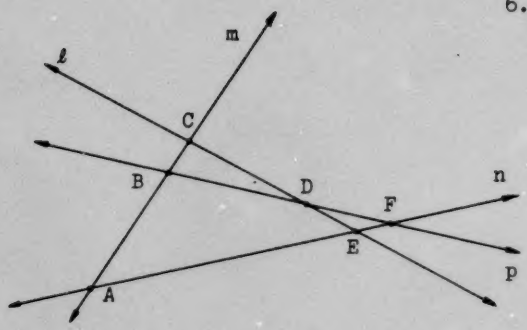
(b) Name the lines you have drawn.

(c) How many such lines are there?

5. Think of 5 points marked on a sheet of paper, no three of which lie on the same line. Now think of all the lines through pairs of these points.

How many such lines are there?

Check your answer with a drawing.



6. Name the point of intersection for the following pairs of lines.

(a) m and p

(b) p and ℓ

(c) ℓ and n

7. For the figure in Exercise 6, name the pairs of lines which have the following points as points of intersection.

(a) A

(b) C

(c) F

1-3. Planes

A plane is an idea in geometry which is suggested by any flat surface such as

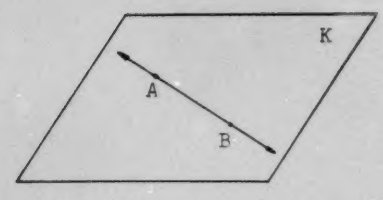
(1) a wall of a room,

(2) the top of a desk,

(3) a door in any position,

(4) a sheet of paper.

A plane is thought of as extending indefinitely in every direction. There are an unlimited number of points in a plane and a plane contains an unlimited number of lines.

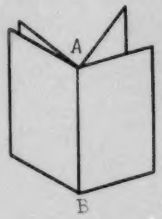


This drawing represents a plane K which contains two points A and B.

Now consider the line $\overleftrightarrow{AB}$. Both the line and the plane extend outward indefinitely. Because of our thinking that all lines are straight and all planes are flat, we have the feeling that the line $\overleftrightarrow{AB}$ and the plane K will never leave each other. Therefore, we shall agree that the following statement is a fundamental characteristic of our space.

Property 2: If two points of a line lie in a plane then all the line lies in the plane.

The figure at the right represents an open notebook. Points A and B mark the endpoints of the binding, and the pages represent planes. This figure suggests that there are many planes which contain any two points.



Property 3: For any two points in space, there are many planes which contain them.

Now consider three points A, B, and C represented by the tips of your thumb and first two fingers spread stiffly as shown so that the points do not lie on the same line.

Take a piece of cardboard representing a plane and place it on the tips B and C of your two fingers. Show that the cardboard may be rotated in different positions while held fixed to the two finger tips. What property does this show?

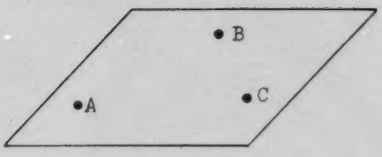
Press the cardboard against all three tips A, B, and C. The position of the cardboard is now fixed. This illustrates another important property.

Property 4: Through any three points, not all on the same line, there is exactly one plane.

This property may also be stated in the following ways.

(1) For any three points in space not all on the same line, there is one plane which contains them and only one plane which contains them.

(2) Any three points not all on the same line determine a plane.



As a result of Property 4, a plane may be named by identifying three of the points in the plane that do not all lie on a line.

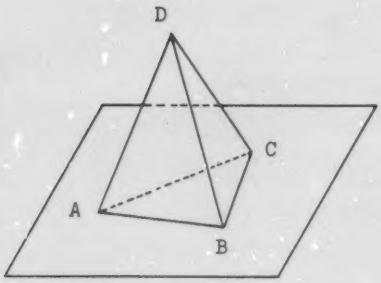
For example, the plane shown at the left may be called "plane ABC" or "plane BAC". The three points may be written in any order.

Exercises 1-3

1. Show how Property 4 helps to explain why a three-legged stool or a tripod always rests flat on the floor.

2. Can a line pass through
   (a) any one point?
   (b) any two points?
   (c) any three points?
   (d) any four points?

3. Can a plane pass through
   (a) any one point?
   (b) any two points?
   (c) any three points?
   (d) any four points?

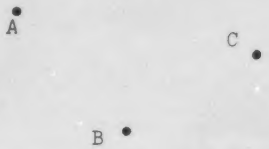
4. 

Points A, B, and C lie in a plane and point D is above the plane.

How many planes are determined by these four points?

Name the planes.

5. Copy the points A, B, C arranged as shown.



(a) Draw a line through points A and B.
Draw a line through points B and C, are $\overline{AB}$ and $\overline{BC}$ in plane ABC?
Give a reason for your answer.

(b) Mark a point R on $\overline{AB}$ and a point S on $\overline{BC}$. Draw $\overleftrightarrow{RS}$. Is $\overleftrightarrow{RS}$ in plane ABC? Give a reason for your answer.

6. (a) Points X, Y, and Z have the property that many different planes contain them. How are points X, Y, and Z arranged? Draw a diagram.

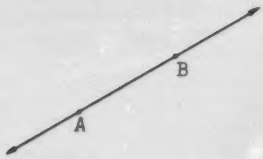
(b) If points X, Y, and Z have the property that they are contained in only one plane, how are points X, Y, and Z arranged? Draw a diagram.

7. How many different lines may pass through

(a) one point?
(b) two different points?

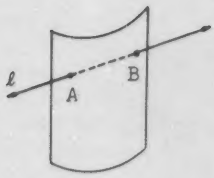
8. How many different planes may pass through

(a) one point?
(b) two points?
(c) three points which are not on the same line?



Explain why there are many planes which may contain a given line $\overleftrightarrow{AB}$.

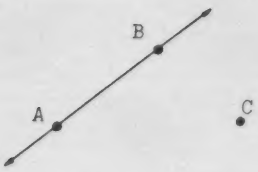
10.



Line ℓ intersects the surface shown at points A and B only.

Explain why the surface could not be a plane.

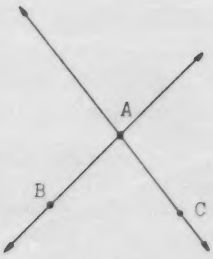
11.



Explain why a line $\overleftrightarrow{AB}$ and a point C not on the line determine a plane.

Name the plane.

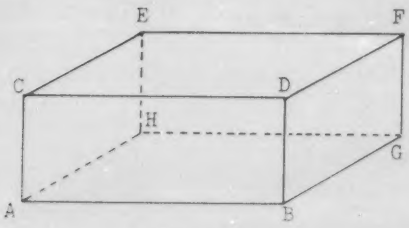
12.



Explain why two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ intersecting at A determine a plane.

Name the plane.

13. The drawing below represents a box. Look at the side of the box determined by points B, G, and F.



(a) Name another point that lies in plane BGF.

Name the plane suggested by

(b) the top of the box,

(c) the bottom of the box,

(d) the front of the box,

(e) the back of the box.

1-4. Intersections

Let A and B be names for the following sets.

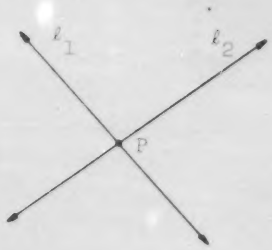
$$A = \{1,3,5,7,9,11\}$$

$$B = \{1,4,9,16,25,36\}$$

The intersection of A and B is the set of all numbers that belong to both A and B. The intersection is the set $\{1,9\}$.

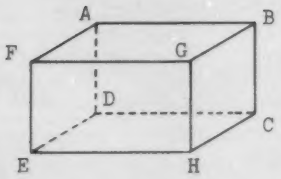
We write: $A \cap B = \{1,9\}$

where "$A \cap B$" means "the intersection of A and B".



The intersection of the lines ℓ_1 and ℓ_2 is the set of all points common to both lines. Since P is the only point in the intersection, we write $\ell_1 \cap \ell_2 = \{P\}$, where "$\{P\}$" means "the set consisting of the point P".

When two lines intersect they have one point in common as was shown from Property 1. When two lines do not intersect they have no points in common.



In the drawing of the box at the left the two lines $\overleftrightarrow{FG}$ and $\overleftrightarrow{EH}$ do not intersect. We say that their intersection is the empty set. The empty set may be named by the symbol "$\emptyset$" or "{}". Therefore $\overleftrightarrow{FG} \cap \overleftrightarrow{EH} = \emptyset$. Since the intersection of $\overleftrightarrow{FG}$ and $\overleftrightarrow{EH}$ is empty and they lie in the same plane, they are said to be parallel. This is written, $\overleftrightarrow{FG} \parallel \overleftrightarrow{EH}$, where the symbol "$\parallel$" means "is parallel to".

The intersection of $\overleftrightarrow{FG}$ and $\overleftrightarrow{HC}$ is also empty, but the two lines do not lie in the same plane. The lines are not parallel but are called skew lines.

When we say that two lines are parallel, we mean:

(1) the two lines lie in the same plane, and

(2) their intersection is the empty set.

Exercises 1-4

1. In each case, write the intersection of the given sets

(a) A = {2,4,6,8,10}
B = {1,4,7,10,13,16}

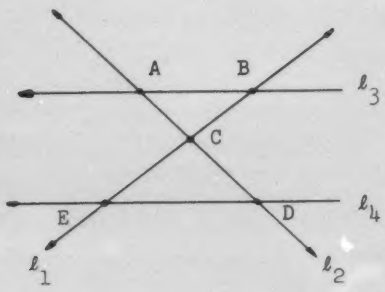
(b) P = {John, Ethel, Bill, Frank, Alice}
Q = {Frank, Paul, Alice, Diane, John, Helen}

(c) The letters used in your full name and the letters used in the name of your school.

(d) The set of national holidays and the set of days in July.

(e) The set of school days in this year and the set of Sundays in this year.

2. In the drawing below, $\ell_3 \| \ell_4$. Write the intersections of the given pairs of sets.



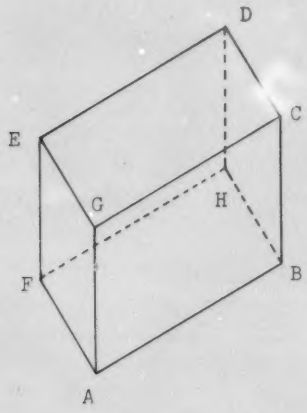
(a) $\ell_1 \cap \ell_3$

(b) $\ell_2 \cap \ell_4$

(c) $\ell_4 \cap \ell_3$

(d) $\ell_1 \cap \ell_2$

3. In the drawing of the box below,



(a) name three lines that are each skew to $\overleftrightarrow{AB}$,

(b) name three lines that are each parallel to $\overleftrightarrow{GC}$.

4. Find the intersections for the following pairs of lines.

(a) $\overleftrightarrow{CD} \cap \overleftrightarrow{HD}$

(b) $\overleftrightarrow{GC} \cap \overleftrightarrow{FA}$

(c) $\overleftrightarrow{BA} \cap \overleftrightarrow{FA}$

(d) $\overleftrightarrow{EH} \cap \overleftrightarrow{BH}$

5. If two lines do not intersect, are they parallel? Explain.

6. Two lines sometimes intersect and sometimes they don't. Also, two lines sometimes lie in the same plane and sometimes they don't.

What do we call two lines which

(a) lie in the same plane and which intersect?

(b) lie in the same plane and do not intersect?

(c) do not lie in the same plane and do not intersect?

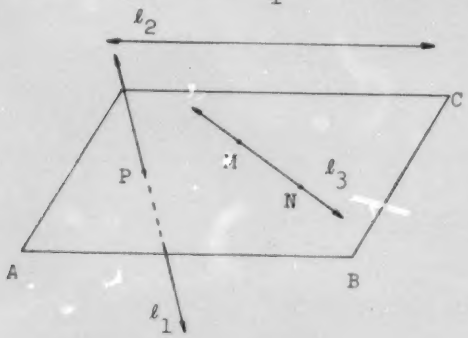
(d) Is it possible for two lines to intersect and not to lie in the same plane?

(e) Copy the table below and write your answers to (a), (b), (c), and (d) in the boxes indicated.

	Intersect	Do not intersect
Lie in same plane	(a) ?	(b) ?
Do not lie in same plane	(d) ?	(c) ?

1-5. Intersections of Lines and Planes

The drawing below shows the possible intersections of a line and a plane ABC. Line ℓ_1 intersects the plane at point P. Then $\ell_1 \cap$ plane ABC = {P}. Line ℓ_2 is parallel to the plane. Therefore $\ell_2 \cap$ plane ABC = $\emptyset$. Two points M and N of line ℓ_3 lie in the plane. Why does ℓ_3 lie in the plane? Explain why the intersection of ℓ_3 and the plane is ℓ_3 itself. Then $\ell_3 \cap$ plane ABC = ℓ_3.



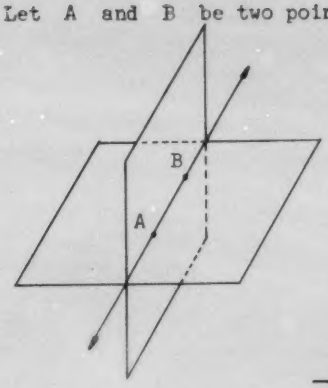
Exercises 1-5a

(Class Discussion)

1. Hold two sheets of paper so that they have only one point of intersection.

   Does this show that two planes may have only one point in their intersection? (Remember that planes extend without limit in all directions.)

2. Let A and B be two points in the intersection of two planes.



(a) Explain why the intersection must contain the entire line $\overleftrightarrow{AB}$.

(b) Explain why if the intersection contains any other point not on $\overleftrightarrow{AB}$ then the intersecting planes are the same plane.

This discussion leads us to another fundamental property in the structure of our space.

Property 5: If the intersection of two different planes is not empty, then the intersection is a line.

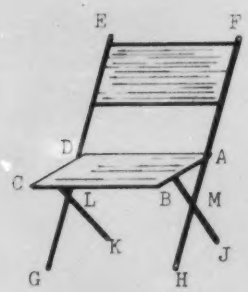
When we say that two planes are parallel, we mean that: their intersection is the empty set.

Exercises 1-5b

1. Describe some physical objects which suggest

(a) two planes intersecting in a line.
(b) a line and a plane intersecting in a point.
(c) two parallel planes.
(d) the intersection of a plane and a line lying in the plane.
(e) a line parallel to a plane.

2. Think of the planes and lines suggested by the figure which represent a folding chair.



Write the following intersections.

(a) $\overleftrightarrow{FH} \cap \overleftrightarrow{BJ}$ =

(b) $\overleftrightarrow{GL} \cap \overleftrightarrow{AD}$ =

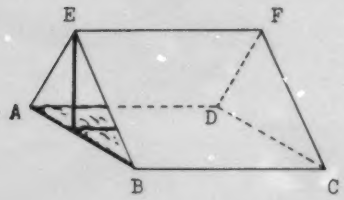
(c) plane ABC $\cap$ $\overleftrightarrow{MJ}$ =

(d) $\overleftrightarrow{BC}$ $\cap$ plane EFA =

(e) plane ABC $\cap$ plane HMJ =

(f) plane CDL $\cap$ plane HMB =

3. Think of the lines and planes suggested by this drawing of a tent.



Name

(a) three planes.

(b) three lines which are skew to $\overleftrightarrow{AB}$.

(c) two lines which are parallel to $\overleftrightarrow{BC}$.

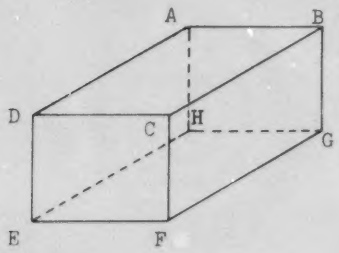
(d) two parallel planes.

(e) two planes whose intersection is $\overleftrightarrow{CD}$.

(f) a line and a plane whose intersection contains the single point E.

(g) a line and a plane that are each parallel to $\overleftrightarrow{AD}$.

4. From the drawing of a box, write the following intersections.



(a) plane EDC ∩ plane FGB =

(b) plane CBG ∩ $\overleftrightarrow{AB}$ =

(c) $\overleftrightarrow{AG}$ ∩ plane FCG =

(d) plane GAB ∩ $\overleftrightarrow{CH}$ =

(e) plane CDG ∩ plane FGB =

(f) plane EDA ∩ plane GBC =

(g) plane FED ∩ $\overleftrightarrow{EC}$ =

5. For the box shown above in Exercise 4, name pairs of planes which have the following lines as lines of intersection.

(a) $\overleftrightarrow{HG}$ (c) $\overleftrightarrow{CE}$

(b) $\overleftrightarrow{FG}$ (d) $\overleftrightarrow{HC}$

6. Describe the three possible positions for a line and a plane, and in each case tell what their intersection is.

7. Describe the two possible positions for two planes and tell what their intersection is in each case.

1-6. Betweenness and Segments

Three boys are standing together. You're told that, "Tom is tall, Sam is short, and Ben is between them."

Can you tell which boy is named Ben?

The word "between" is confusing here. Ben could be between Sam and Tom in height, or he could be between them in position. The best we could do is guess which boy is Ben.

The language of mathematics can not be so confusing that it forces us to guess about the meaning of a statement. We need to know exactly what it means to say "point B is between points A and C". We would probably all agree that the phrase should not refer to the alphabetical order of the points because the letters are only names for the points. However, it should have something to do with some kind of order.

We will use the word "between" for points only when the points in question are on the same line. Look at points P, Q, R, and S:

Figure 1

R is on the line $\overleftrightarrow{PQ}$ and the order of the points is P, R, Q (or Q, R, P) so we can say "R is between P and Q". However, point S is not on line $\overleftrightarrow{PQ}$, so it is not between P and Q. Look at the points A, B, X, and Y on line AY:

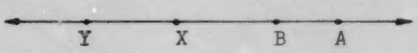


Figure 2

Is B between A and X?
Is X between Y and A?
Is A between B and Y?

Your answer to the third question should be "No" because the order of the three points involved is Y, B, A (or A, B, Y) and that means point B is the one in between.

In Figure 2 we know that X is between A and Y and that B is also between A and Y. In fact, there are many points between A and Y. We haven't named them all because we couldn't.

When we say that a point P is between points A and B, we mean exactly two things:

(1) There is a line containing A, B, and P;

(2) On that line, the points are in the order A, P, B or B, P, A.

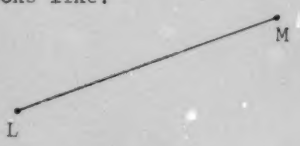
We can simplify this definition a little by using the word "collinear". A set of points is collinear if there is a line which contains all the points of the set. For example, in Figure 1 the set of points containing P, Q, and R is collinear, while points P, Q, R, and S are not collinear. Using this word, we may restate our definition:

When we say that point P is between points A and B, we mean:

(1) A, B, and P are collinear, and

(2) on their line the points are in the order A, P, B or B, P, A.

Think of two different points L and M. Can you count all the points between L and M? Think of the two points L and M together with all the points between them. The set of points you are thinking of is a part of a line, and a picture of it looks like:

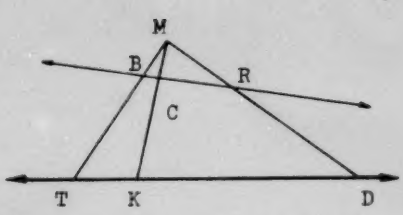


Such a set of points is called a line segment, or more simply, a segment. Points L and M are the endpoints of the segment. We name the segment by its endpoints; for example, the segment shown is named $\overline{LM}$ or $\overline{ML}$.

Definition. The segment $\overline{AB}$ is the set of points consisting of the two points A, B, and all points between A and B.

Exercises 1-6a

1. In the figure at the right,



   (a) is M between B and R?
   (b) is C between B and R?
   (c) is C between M and K?
   (d) is C on segment $\overline{MK}$?
   (e) Name 3 segments on $\overleftrightarrow{TD}$.
   (f) Are T, K, and R collinear?
   (g) Is K between T and R?
   (h) Name 4 sets of 3 collinear points.

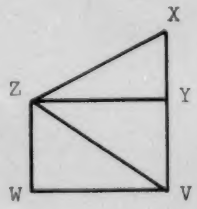
2. (a) Draw a picture representing 3 segments, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, where points A, B, and C are <u>not</u> collinear.

   (b) What would your picture represent if A, B, and C were collinear?

3. What word in the definition of segment shows that a segment is part of a <u>line</u>? Explain.

4. Points A, B, and C are collinear.
   If the distance from A to B is 5 inches, and
   the distance from B to C is 12 inches, and
   the distance from A to C is 7 inches,
   draw a diagram showing A, B, and C in their correct order. Which point is between the other two?

5. In the figure at the right,



   (a) Name 4 segments that intersect at Z.
   (b) How many segments have Y as an endpoint?
   (c) There are 4 segments which contain point Y. Name them.
   (d) What is $\overline{WZ} \cap \overline{WV}$?

6. Draw two segments $\overline{AB}$ and $\overline{CD}$ for which $\overline{AB} \cap \overline{CD}$ is empty but $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is one point.

7. Mark two different points, X and Y.

(a) Draw some segments which contain X and Y.

(b) How many segments could contain X and Y?

(c) How many lines could contain X and Y?

(d) How many segments have both X and Y as endpoints?

(e) We have agreed in Property 1 that "through any two different points there is exactly one line". Is it true that "through any two different points there is exactly one segment"?

BRAINBUSTER:

Points P, Q, and R are collinear. The distance from P to Q is greater than the distance from Q to R, and the distance from Q to R is greater than the distance from P to R.

Which point must be between the other two?

1-7. Separation

Mark any point A on a sheet of paper, and place the tip of your pencil on that point. Trace out any drawing you like without lifting your pencil from the paper. Your drawing might look like any one of these:

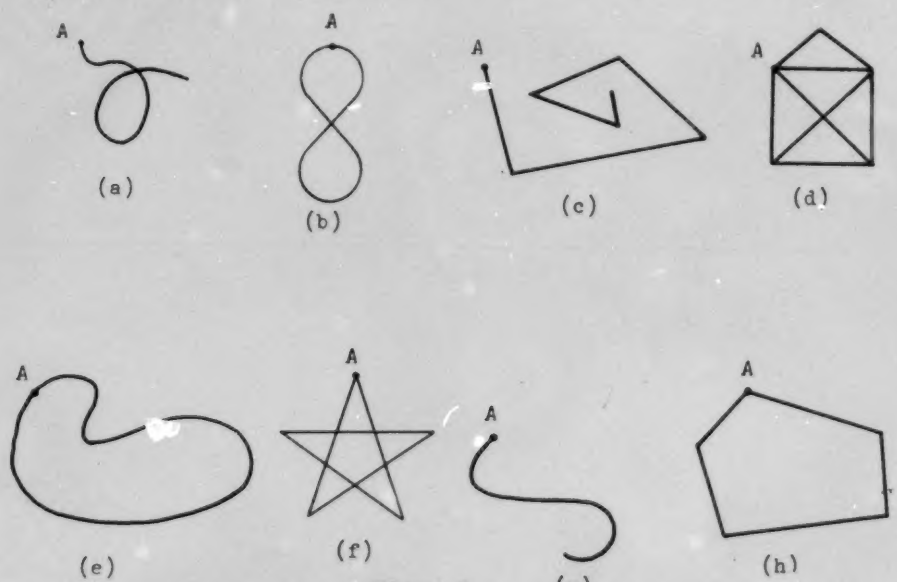


Figure 1

The geometric idea which all of these pictures represent is called a path. The paths you drew were contained in the plane of your paper. It is possible to think of paths in space, but in this section we will work only with paths in a plane.

Look at the paths in Figure 1. Some of them cross themselves, like paths (a), (b), and (f). Some paths do not cross themselves, like (c) and (g). We will call a path which does not cross itself a simple path.

Some paths return to their starting points, like (b), (e), (f), and (h). We will call a path which does not cross itself and which goes back to its starting point a simple closed path. Which of the paths in Figure 1 are simple closed paths?

What do you suppose a simple open path looks like? Such a path does not cross itself and does not go back to its starting point. Figures (c) and (g) are examples of simple open paths.

You can invent names for the other special kinds of paths, but we will only be concerned with these two kinds of simple paths.

A simple closed path has an interesting property which no simple open path has. Draw a simple closed path, perhaps something like path p in Figure 2. Mark two points A and B inside the simple closed path, and mark a point C outside.

Figure 2

Try to draw a path in the plane from A to B without intersecting p. Try to draw a path from A to C without intersecting p. Finally, try to draw a path from B to C without intersecting p. Were you able to draw all three paths, or were there situations in which you could not?

If you tried to draw all three paths, you probably noticed that a simple closed path like p in Figure 2 has the property that it can separate points in the plane. We say that points A and C, for example, are separated by path p because (1) A and C are not on path p, and

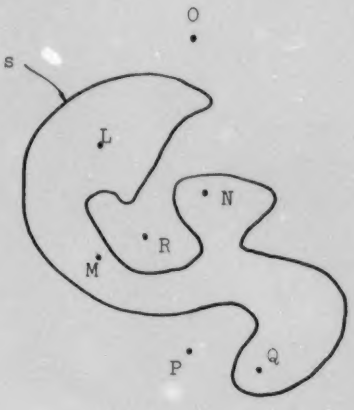
(2) every path connecting A and C intersects p. Notice, however, that points A and B are not separated by path p; you should have been able to draw a path connecting them without intersecting p.

Are points B and C separated by path p in Figure 2?

Exercises 1-7a

(Class Discussion)

1. The figure at the right shows a simple closed path s and points L, M, N, O, P, Q, and R not on the path s.



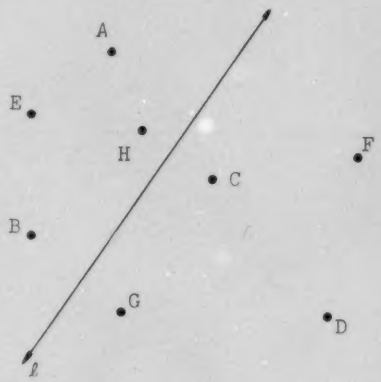
   (a) Name the points which are separated from point N by path s.

   (b) Name the points which are not separated from N by s.

   (c) Trace the drawing, and then draw paths on your tracing which connect point N to the points you named in (b).

   (d) Make another tracing of the drawing, and on it shade all the points on the page which are not separated from point N by path s. This set of points is called the interior of s.

2. The figure at the right shows a line ℓ and points A, B, C, D, E, F, G, and H not on ℓ.



   (a) Which of the points named can be connected to point H by paths which do not intersect ℓ?

   (b) Which of the points named are separated from H by ℓ?

   (c) Imagine all the points in the plane which are separated from H by ℓ. Copy the figure above, and shade the part of the plane separated from H by ℓ.

3. Show why points X and Y are not separated by the simple open path connecting A and B.

4. Show why a segment does not separate any points in the plane in which it lies.

A line separates points in a plane just as a simple closed path does. Suppose line ℓ separates points A and B in some plane.

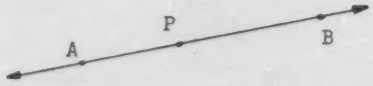
The line ℓ separates the plane into two sets of points:

All the points not on ℓ in the plane which are separated from point A by ℓ, and

All the points not on ℓ in the plane which are separated from point B by ℓ.

Each of these sets of points is called a half-plane. Together they form a pair of opposite half-planes. Line ℓ is the edge or boundary of the two half-planes. We sometimes name a half-plane by one of its points; for example, the "B half-plane" contains, among many others, point B. When there wouldn't be any confusion, we can call the B half-plane determined by line ℓ the "B-side of ℓ", and opposite half-planes determined by ℓ can be called "opposite sides of ℓ".

Just as a line separates a plane into two half-planes, a point separates a line into two half-lines.



The point P separates the line into two sets of points:

All the points except P on the line which are separated from point A by P, and

All the points except P on the line which are separated from point B by P.

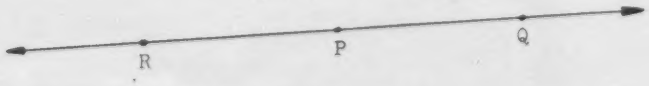
Each of these sets is called a half-line. Together they form a pair of opposite half-lines. Point P is the boundary of the two half-lines.

Point P is not contained in either of the half-lines it determines. If we join point P to one of its half-lines, the resulting set of points is called a ray.

Definition. A ray is a half-line together with its boundary point. The boundary point is called the endpoint of the ray.

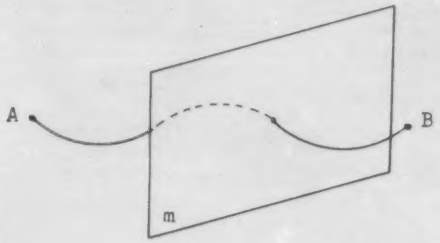
If P is the endpoint of a ray and Q is another point of the ray, we name the ray "$\overrightarrow{PQ}$". Note that $\overrightarrow{PQ}$ is not the same as $\overrightarrow{QP}$. We now have a geometric meaning for the term "ray" which should match very well the picture we have of a "ray of light".

Two opposite half-lines, each joined with the common boundary point, form a pair of opposite rays. For example,



$\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are opposite rays; point P is their common endpoint. Note that it is always true that if P is between Q and R, PQ and PR are opposite rays, and $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ together form a line.

It should now come as no surprise that a plane separates space into two half-spaces. We will assume that points A and B in space are separated by plane m if each path from A to B in space intersects m.



In the figure, plane m separates space into two sets of points:

All the points not on m in space which are separated from point A by m, and

All the points not on m in space which are separated from point B by m.

Each of these sets is called a half-space. Plane m is the boundary of the two half-spaces.

Exercises 1-7b

1. We now know that

(I) a simple closed path separates a plane
(II) a point separates a line
(III) a line separates a plane
(IV) a plane separates space.

Each of the following is a physical example of one of these separation ideas. For each, write the Roman Numeral of the separation idea which the example suggests to you.

(a) the net on a tennis court
(b) a shut-off valve on a pipeline
(c) a moat around a castle
(d) a dividing line down the middle of a road

(e) a station on a railroad line

(f) a window in your living room

(g) the plane of a movie screen

(h) the shore of a lake

(i) the Mason-Dixon line

2. Draw a line ℓ and mark points A and B on opposite sides of ℓ.

(a) Will <u>every</u> path (in the plane) from A to B intersect ℓ?

(b) Draw $\overline{AB}$. Does it intersect ℓ? (Can $\overline{AB}$ be thought of as a path?)

(c) Mark points C and D so that $\overline{AC}$ and $\overline{AD}$ intersect ℓ. On which side of ℓ must you mark C and D?

(d) Mark point E so that $\overline{AE}$ does <u>not</u> intersect ℓ. On which side of ℓ must you mark E?

(e) Using parts (b) and (c), state a simple test for showing that some points P and Q are on <u>opposite</u> sides of ℓ.

(f) Using part (d), state a simple test for showing that some points R and S are on the <u>same</u> side of ℓ.

(g) Would your tests in (e) and (f) still work if ℓ looked "wiggly"?

3. Not counting the lines themselves, into how many sets of points is a plane separated by two lines in the plane

(a) if the lines are parallel?

(b) if the lines intersect?

4. Tell whether each of the following statements is true or false. Draw a diagram supporting your answer.

(a) A segment contains many other segments.

(b) Each segment contains two rays.

(c) The words "ray" and "half-line" have the same meaning.

(d) A half-plane contains many rays.

(e) A ray contains many other rays.

(f) A half-plane contains many other half-planes.

5. In each case, identify the boundary between

   (a) two half-lines.
   (b) two half-planes.
   (c) two half-spaces.

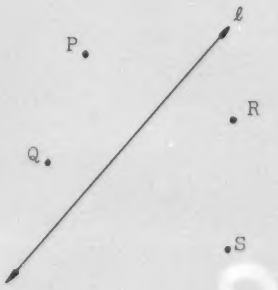
6. Tell whether each of the following is true or false.

   (a) Space is separated by a half-plane.
   (b) Space is separated by a line.
   (c) Space is separated by a ray.
   (d) Space is separated by a closed box.
   (e) Space is separated by an open box.
   (f) Space is separated by an inflated volley ball.
   (g) Space is separated by a point.

7. Not counting the planes themselves, into how many sets of points is space separated by two planes,

   (a) if the planes are parallel?
   (b) if the planes intersect?

8. In the figure at the right, line ℓ and points P, Q, R, and S are in one plane. Are the following statements true or false?



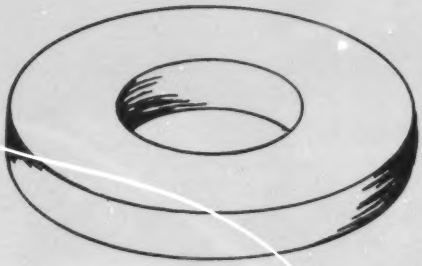
   (a) The R-side of ℓ is the same as the S-side of ℓ.
   (b) The S-side of ℓ is the same as the P-side of ℓ.
   (c) $\ell \cap \overline{PQ}$ is empty.
   (d) $\ell \cap \overleftrightarrow{PQ}$ is empty.
   (e) $\ell \cap \overrightarrow{PQ}$ is empty.

9. Draw a horizontal line. Label four points on it, A, B, C, and D in that order from left to right. Name two rays

   (a) which are opposite.
   (b) whose intersection contains exactly one point.
   (c) whose intersection is empty.
   (d) whose intersection is segment $\overline{CD}$.

BRAINBUSTER #1

The geometric idea which is a model for the surfaces of shapes like doughnuts, tire innertubes, or candy Life Savers is called a torus.



Find a simple closed path on a torus which does not separate any points on the surface of the torus.

BRAINBUSTER #2

A child wandered off from his parents in a park that was fenced, but had several gates. Guards at the gates reported he had passed through their gates as follows:

Gate #1 -- 3 times
Gate #2 -- 7 times
Gate #3 -- 5 times
Gate #4 -- 7 times

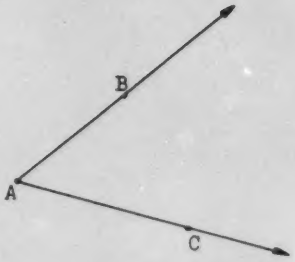
Where would you look for the child?

1-8. Angles

Angles have many applications in our modern civilization. An airplane pilot and a ship navigator use angles to chart their courses. An engineer builds a road so that the angle that the road makes with the horizontal plane is not too steep. A builder makes sure that the plane of a wall and the plane of the floor meet at right angles. In this section, we will review the concept of angle and some of the ideas that are related to it. However, we first need to recall the idea of the "union" of two sets.

Definition. The union of two sets of points is the set consisting of all the points in those sets.

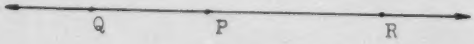
For example, the figure at the right represents the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ because it consists of all the points in $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Using the symbol "$\cup$" to stand for "union", we can write "$\overrightarrow{AB} \cup \overrightarrow{AC}$" to stand for the figure at the right, the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



Furthermore, the figure shown also represents what we mean by the word "angle". Therefore, we can say:

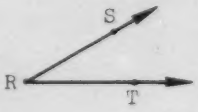
> Definition. An angle is the union of two non-collinear rays having the same endpoint.

Why "non-collinear"? Remember that sets of points are collinear if there is a line which contains them. Therefore, if we allowed collinear rays to form an angle,

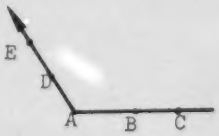


then the line formed by opposite rays $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ would have to be thought of as a kind of "angle". We will agree for now that no angle can be the same as a line, and hence, that any angle is made up of rays which are not on the same line, that is, "non-collinear" rays. (Perhaps the class discussion exercises will give you further reason for wanting the definition of angle to exclude collinear rays.)

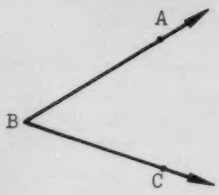
In the figure at the right, we have a representation of an angle. Point R, the common endpoint, is the vertex of the angle, and $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are the sides of the angle. In order to be able to talk about angles we must agree on a method of naming them.



The angle illustrated at the right might be called $\angle EAB$, or $\angle EAC$, or $\angle BAD$, or $\angle DAC$. Note that the letter at the vertex is always written between the letters on the rays. In cases where no confusion is likely to result we name the angle by naming the vertex. In this case, we would refer to $\angle A$.



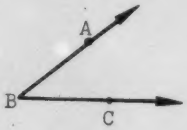
Do you think that every angle is contained in exactly one plane? Let us see why this must be so. Since points A, B, and C do not lie on the same line (they are not collinear) they must lie in exactly one plane by Property 4. Thus, we may talk about <u>the plane</u> of an angle.



<u>Exercises 1-8a</u>

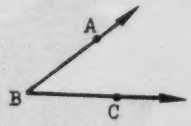
(Class Discussion)

1. Draw an angle ABC.

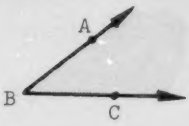


2. Does the angle separate the plane? If so, how can we describe the parts of the plane?

3. Shade the portion of the plane that appears outside the angle, as shown. This shaded portion is called the <u>exterior</u> of the angle.



4. Shade the portion of the plane that appears inside the angle, as shown. This shaded portion is called the <u>interior</u> of the angle.



5. (a) Describe the A-side of $\overleftrightarrow{BC}$.
   (b) Draw a new $\angle ABC$ and shade the A-side of $\overleftrightarrow{BC}$.

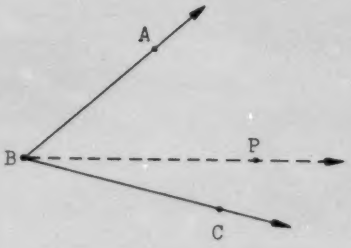
6. (a) Describe the C-side of $\overleftrightarrow{AB}$.
   (b) On the figure you drew in part 5(b), shade in a different way the C-side of $\overleftrightarrow{AB}$.

7. Describe the doubly-shaded region in your drawing in 6(b).

These exercises suggest the following definition:

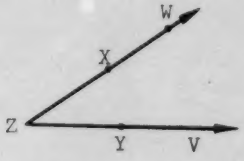
Definition. The interior of $\angle ABC$ is the intersection of the A-side of $\overleftrightarrow{BC}$ and the C-side of $\overleftrightarrow{AB}$.

If point P is in the interior of $\angle ABC$ we say that $\overrightarrow{BP}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Note that betweenness for rays is similar to but not identical to betweenness for points.



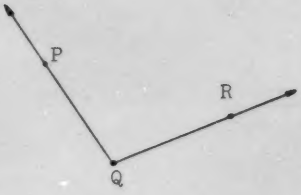
Exercises 1-8b

1. In the figure
   (a) name $\angle Z$ in two other ways.
   (b) name the vertex of $\angle Z$.
   (c) name the rays of $\angle Z$.



2. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$.
   (a) Shade the C-side of $\overleftrightarrow{AB}$.
   (b) Shade the A-side of $\overleftrightarrow{BC}$.
   (c) Describe the set that is doubly shaded.

3. For the figure at the right, write another name for
   (a) $\overrightarrow{QP} \cup \overrightarrow{QR}$
   (b) $\overrightarrow{QP} \cap \overrightarrow{QR}$

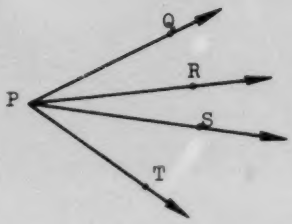


4. Tell whether each of the following is true or false.
   (a) The vertex of an angle is in the interior of the angle.
   (b) If two rays are not on the same straight line they form an angle.
   (c) An angle is the union of its interior and its exterior.
   (d) An angle separates its plane.
   (e) A point is in the interior of an angle if it is on a line which intersects the sides of the angle.

5. (a) Name three angles that have $\overrightarrow{PS}$ as a side.

(b) What is $\overrightarrow{PQ} \cup \overrightarrow{PS}$?

(c) What is the intersection of the interior of $\angle QPS$ and the interior of $\angle RPS$?



6. Suppose $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$, and $\overrightarrow{OX}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Then

(a) $\overrightarrow{OB}$ is also between ___?___ and ___?___.

(b) $\overrightarrow{OX}$ is also between ___?___ and ___?___.

7. Draw $\angle ABC$.

(a) Draw a few rays between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

(b) Imagine all such rays drawn. How can you describe the figure formed?

8. Draw $\angle ABC$.

(a) In plane ABC draw a few rays with endpoint B that are not between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

(b) Imagine all such rays drawn. How can you describe the figure formed if $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are excluded?

Exercises 1-8c

(Class Discussion)

1. (a) Points X, Y, and Z are non-collinear.
Draw a figure representing the union of $\overline{XY}$, $\overline{YZ}$, and $\overline{XZ}$.

(b) Points L, M, and N are collinear.
Draw a figure representing the union of $\overline{LM}$, $\overline{MN}$, and $\overline{LN}$.

(c) The figure in part (a) is called a triangle, and we name it triangle XYZ or "$\triangle XYZ$". Is $\triangle XYZ$ a plane figure? How do you know?

(d) Using part (a), write a definition of a triangle beginning "Triangle XYZ is the union of ...".

(e) Is a triangle a simple closed path?

2. Mark 3 non-collinear points A, B, and C.
Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$.

(a) Shade the A-side of $\overleftrightarrow{BC}$.

(b) Shade the B-side of $\overleftrightarrow{AC}$. Name the set of points which is doubly shaded.

(c) Shade the C-side of $\overleftrightarrow{AB}$. The set of points which is shaded three times is called the interior of $\triangle ABC$.

(d) Using parts (a), (b), and (c) and the idea of intersection of sets, write a definition for the interior of $\triangle ABC$, beginning "the interior of $\triangle ABC$ is the intersection of ...".

(e) Pick some point P in the interior of $\triangle ABC$. Pick another point Q which is not in the interior of $\triangle ABC$ and which is not on the triangle either. Does $\overline{PQ}$ intersect $\triangle ABC$? Always?

Definition. For any three non-collinear points A, B, and C, the union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called a triangle.

1-9. Locating Positions and Points

Below is shown a seating chart for a class in which the rows, and the seats in each row are numbered.

5	Kay	Mike	Nora	Eve	Carl
4	Fred	Pete	Gary	May	Nell
3	Ray	Ed	June	Myra	Paul
2	Ann	Emma	Bill	Don	Max
1	John	Mary	Jane	Jim	Sue
SEAT / ROW	1	2	3	4	5

Figure 3

From the chart we see that Ed sits in Row 2, Seat 3. This could be shortened to (R2,S3).

If we agree that the number for the row is named first and the seat number is named second then Ed's position is located by the pair of numbers (2,3). In this way a pair of numbers can be used to locate a position in a plane.

With the same agreement in naming the row and seat, Bill's position is located by the pair (3,2). Therefore, the pair (2,3) and the pair (3,2) locate different positions. Since the order of the pair numbers is important in describing a position, we shall agree to the order, (row, seat) for this discussion.

Exercises 1-9a

(Class Discussion)

1. (a) Who sits at (4,5)?
   (b) Who sits at (5,4)?
   (c) What pair of numbers gives Pete's position?
   (d) What pair of numbers gives Don's position?

2. Write the set of all the number pairs which correspond to seats occupied by girls.

3. Consider the set of students:

   {Fred, Pete, Gary, May, Nell}.

   (a) What do their positions have in common?
   (b) Write the set of number pairs corresponding to these students.
   (c) What do these number pairs have in common?

4. Write the set of number pairs corresponding to the set of students:

   {Jane, Bill, June, Gary, Nora}.

   State what the number pairs have in common and what the positions of these students have in common.

5. (a) What is the intersection of the sets of students in Exercises 3 and 4?
   (b) What is the intersection of the sets of number pairs in Exercises 3 and 4?

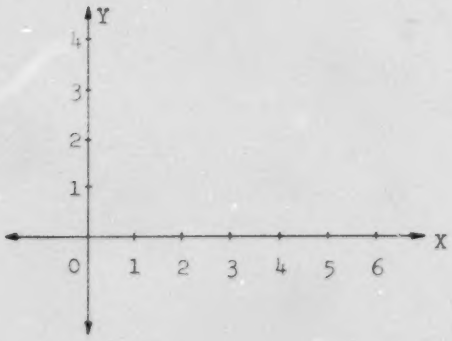
6. (a) Find the set of students having row number and seat number equal.
   (b) Find a geometric way of describing the positions of these students.

7. (a) Find the set of students having their seat number greater than their row number.
   (b) How can you express their positions geometrically?

The idea of using a pair of numbers to locate a position on a seating chart can be applied to locating points in a plane.

First, draw two lines, one horizontal and one vertical.

Notice that the four angles determined by these lines are all right angles. The point of intersection of the two lines is to be the zero-point (origin) on both lines. Using the same unit of distance on both lines, we can mark off points on each line at equal intervals.



The horizontal line is called the X-axis and the vertical line is called the Y-axis. The plural of the word "axis" is "axes" (pronounced ack-seeze, with the accent on the ack). The point of intersection of the axes is called the origin.

Now let's see how a point such as S in Figure 2 below can be located.

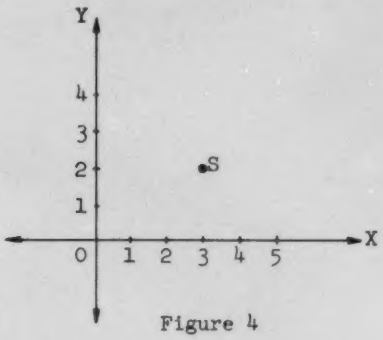


Figure 4

First, draw a vertical line ℓ_1 through S as in Figure 5.

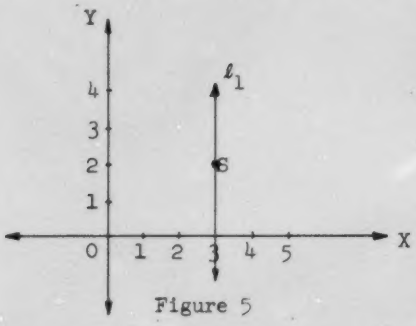


Figure 5

Look at the point where ℓ_1 crosses the X-axis. In this case, the vertical line crosses the X-axis at 3. Now look at Figure 6, where a horizontal line ℓ_2 has been drawn through S.

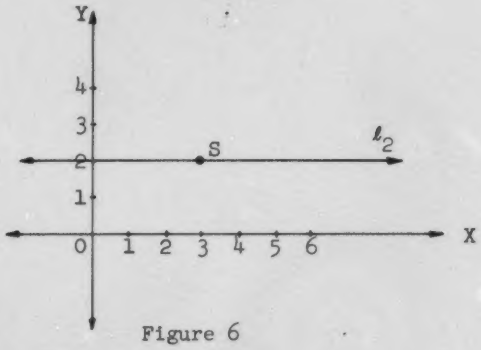


Figure 6

Here, the horizontal line crosses the Y-axis at 2. We can now locate S by the vertical line through 3 on the X-axis and by the horizontal line through 2 on the Y-axis. With the agreement that the number on the X-axis is to be named first, S is then located by the number pair (3,2). If we denote a number on the X-axis by the letter x and a number on the Y-axis by the letter y, we can write

$$(x,y) = (3,2).$$

Thus, we have a way of naming various points in the plane.

Drawing a point in the plane when the number pair is given for it is called <u>plotting</u> the point. To plot a point, we simply reverse the process described above. For example, if we were to plot (5,3), we draw the vertical line through 5 on the X-axis and the horizontal line through 3 on the Y-axis. Where these two lines intersect is the point (x,y) = (5,3).

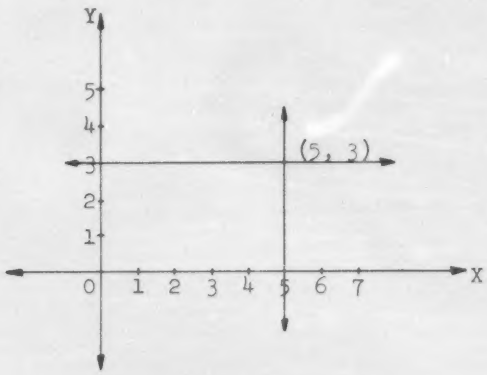
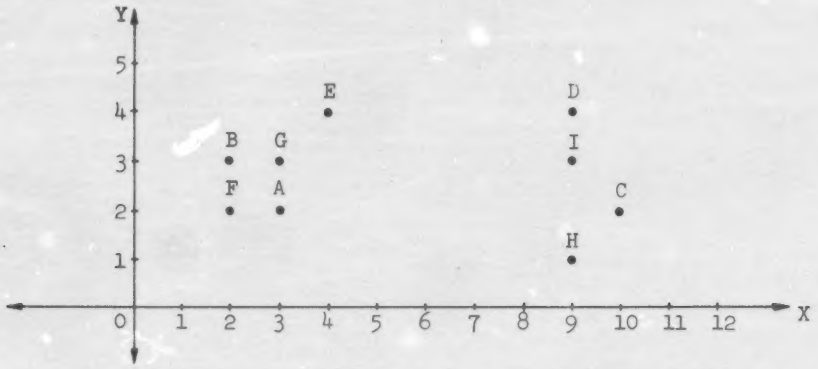


Figure 7

Exercises 1-9b

1. Using the picture below, fill in the table.



Point	A	B	C	D	E	F	G	H	I
x	?	?	?	?	?	?	?	?	?
y	?	?	?	?	?	?	?	?	?

2. Give the number pair for each of the points in Exercise 1.

Point	A	B	C	D	E	F	G	H	I
Number Pair	?	?	?	?	?	?	?	?	?

3. (a) What do points B, G, and I have in common?
   (b) What do the number pairs for B, G, and I have in common?

4. (a) What do points E, F, and G have in common?
   (b) Why might you say that the point (91,91) would have something in common with E, F, and G?
   (c) How might the point (91,91) be related geometrically to E, F, and G?

5. (a) What do points D, H, and I have in common?
   (b) If a point (x,59) has the same common property, what is the number x?

6. Plot the points located by the number pairs given below.

Point	P	Q	R	S	T	U
Number Pair	(1,4)	(6,1)	(4,4)	(0,0)	(2,5)	(0,5)

Point	V	W	X
Number Pair	(3,4)	(5,2)	(4,0)

7. (a) What do points Q, T, V, and W have in common?
   (b) If (1,y) has the same common property, what is y?

8. Give a number pair for a point between P and V that has the same common property shared by P, R, and V.

9. How can you tell by the number pair whether a point is on the X-axis?

Thus far, we have been able to locate points that can be named by pairs of whole numbers.

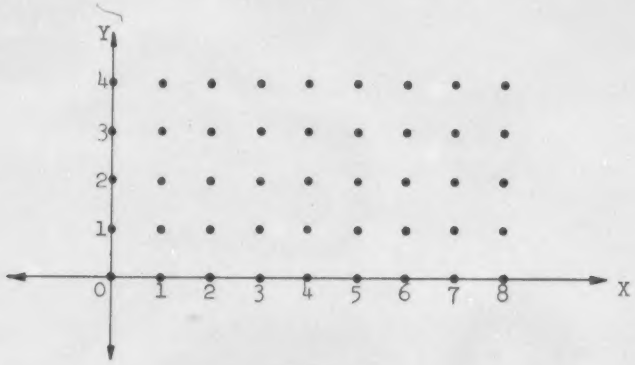


Figure 8

There are many more points which can be located in the part of the plane shown in Figure 8 by using such pairs of numbers as $(\frac{3}{2}, \frac{9}{13})$, $(\frac{17}{5}, \frac{22}{3})$ and so on. In this way more and more points of this part of the plane can be filled in until our picture begins to look like the following.

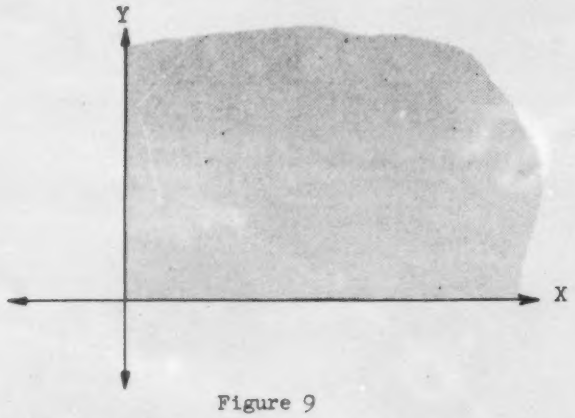


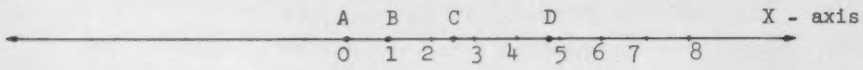
Figure 9

Notice that this still gives us only part of the plane. No point to the left of the Y-axis nor any point below the X-axis has been identified.

In the next sections we shall develop a way of locating and naming these other points.

1-10 Coordinates

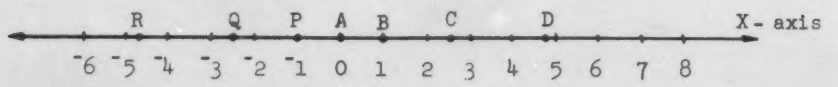
In constructing the horizontal number line called the X-axis, the point of intersection with the Y-axis (point A below) was assigned the number 0. A unit point (point B below) was selected and assigned the number 1. Then points at unit intervals were marked along the ray $\overrightarrow{AB}$ and assigned numbers as shown below.



You will see eventually that each point on ray $\overrightarrow{AB}$ can be assigned a number, as for example, the number for C is $2\frac{1}{2}$ or $\frac{5}{2}$ and for D is 4.8. The numbers assigned to the points on the line are called coordinates of the points. The coordinate of point B is 1 and the coordinate of D is 4.8 on the X-axis above. What is the coordinate of A? of C?

Now consider the opposite ray of ray $\overrightarrow{AB}$ on the X-axis. Mark a point (point P below) at a unit interval to the left of A. The coordinate of point P is $^{-}1$ (read "negative one") which is the "opposite" of the number 1.

The coordinate of the next interval point is $^{-}2$ (negative 2) which is the "opposite" of 2. This process is continued as shown below.



In this way numbers are assigned to each point on ray $\overrightarrow{AP}$, as for example, the coordinate of point Q is $^{-}2\frac{1}{2}$ or $-\frac{5}{2}$ which is the opposite of the coordinate of point C. What is the coordinate of point R if it is the opposite of the coordinate of point D?

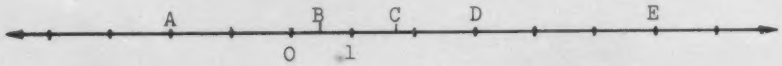
The numbers assigned to the points on half-line AP are called <u>negative numbers</u>, and the numbers assigned to points on half-line AB are called <u>positive numbers</u>. The number 0 is neither positive nor negative.

The Y-axis may be "coordinatized" in the same way as was just shown for the X-axis.

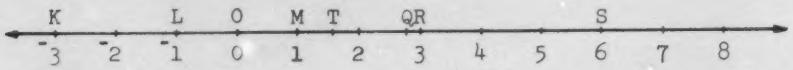
<u>Exercises 1-10</u>

1. Describe how positive and negative numbers may be assigned as coordinates of points on the Y-axis.

2. Referring to the diagram below, give the coordinates of each of the points E, A, B, and C, if C is located $\frac{2}{3}$ the distance from 1 to 2, and if B is midway between points with whole numbers for coordinates.



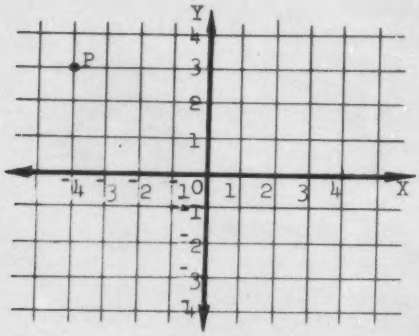
3. Give the letter name in the diagram below for the point whose coordinate is each of the following: $\frac{3}{2}$, $^{-}3$, 2.8, $^{-}1$, 0.



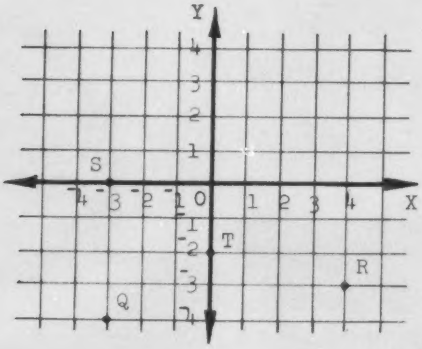
1-11. Coordinates in the Plane

Now that we have a way of assigning both positive and negative numbers (coordinates) to points on the X-axis and Y-axis, we can continue with locating points in various parts of the plane.

In the drawing below the horizontal and vertical lines are shown to help locate points. The vertical line through P crosses the X-axis at $^{-}4$ and the horizontal line through P crosses the Y-axis at 3.



Then point P is located by the pair $(^{-}4,3)$ where $^{-}4$ is called the X-coordinate and 3 is called the Y-coordinate of point P. Notice that the X-coordinate of P is negative and the Y-coordinate of P is positive.

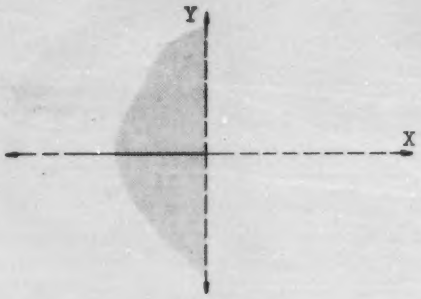


For point Q, you can tell at a glance that both coordinates are negative. Its coordinates are $(^{-}3,^{-}4)$.

For point R, which coordinate is positive and which is negative? The coordinates for R are $(4,^{-}3)$.

The Y-coordinate for S is 0. Why? Its coordinates are $(^{-}3,0)$. Which coordinate of T is 0? The coordinates of T are $(0,^{-}2)$.

A plane in which each point is assigned a pair of coordinates and each pair of coordinates designates a point is called a coordinate plane. We can tell in which part of the coordinate plane a point lies by glancing at its coordinates. For example, let P be a point which has a negative X-coordinate and a positive Y-coordinate.

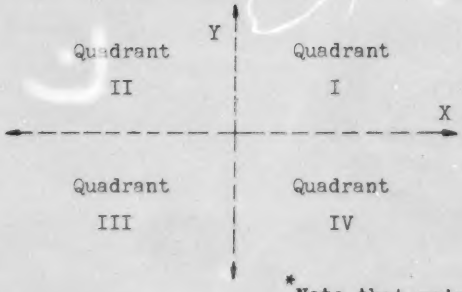


All points with negative X-coordinates lie in the half-plane to the left of the Y-axis, as shaded in the diagram.

All points with positive Y-coordinates lie in the half-plane above the X-axis as shaded in the diagram.

All points with negative X-coordinates and positive Y-coordinates lie in the intersection of the two half-planes.

Therefore, P lies in the part of the coordinate plane which is shaded in the diagram to the left.

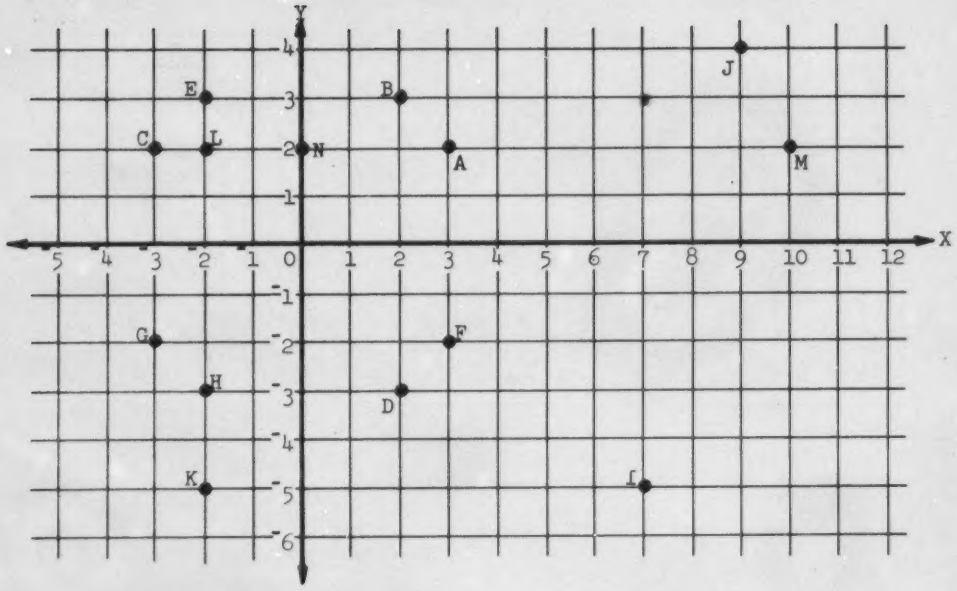


The four intersections of the half-planes for the X- and Y-axes are called quadrants. They are labeled with Roman numerals I, II, III, and IV as shown at the left.

*Note that points on the X- and Y-axes are in none of the quadrants.

Exercises 1-11

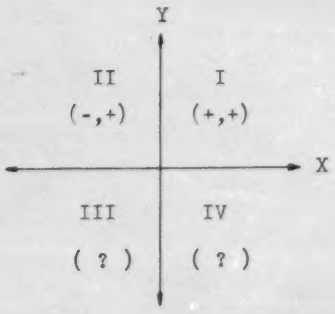
1.



Using the diagram above, fill in the following table.

Point	A	B	C	D	E	F	G	H	I	J	K	L	M	N
Coordinates	?	?	?	?	?	?	?	?	?	?	?	?	?	?

2. For the diagram below, in Quadrant I, the symbol "(+,+)" means that the points are both positive.



In Quadrant II, the symbol "(-,+)" means that the points have negative X-coordinates and positive Y-coordinates.

Write the symbols for the coordinates of points in Quadrant III and Quadrant IV.

3. Plot the following points and draw line segments connecting A to B, to C, ..., and finally, from P to A. [The notation, A(0,9) means that A is a letter used to name the point whose coordinates are (0,9).]

A (0,9), B (5,5), C (2,6), D (6,2), E (2,3).
F (7,$^-$2), G (1,$^-$2), H (1,$^-$7), I ($^-$1,$^-$7), J ($^-$1,$^-$2),
K ($^-$7,$^-$2), L ($^-$2,3), M ($^-$6,2), N ($^-$2,6), P ($^-$5,5).

4. Do the same as for Exercise 1 for the points listed below.

A (6,1), B (8,4), C (7,0), D (10,$^-$3), E (6,$^-$1),
F ($^-$3,$^-$5), G ($^-$9,$^-$4), H ($^-$6,$^-$2), I ($^-$9,$^-$2),
J ($^-$7,1), K (0,4), L ($^-$1,2).

5. (a) Plot the following points: A (7,$^-$3), B ($^-$2,10), C ($^-$2,$^-$6), D (7,7), E ($^-$8,2).

(b) Draw the line segments: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EA}$.

6. Without plotting, give the quadrants of the points whose coordinates are given below.

(a) (3,5)
(b) (1,$^-$4)
(c) ($^-$4,$^-$4)
(d) ($^-$3,$^-$1)
(e) (8,6)
(f) ($^-$7,1)
(g) ($^-$3,$^-$5)
(h) (0,2)
(i) (31,$^-$31)
(j) ($^-$7,$^-$72)
(k) ($^-$51,25)
(ℓ) ($^-$27,$^-$100)

7. Plot all of the following points on one coordinate plane.

(a) A ($\frac{7}{4}$, 2)
(b) B ($-\frac{1}{2}$, $-\frac{1}{2}$)
(c) C ($\frac{10}{3}$, $-\frac{4}{3}$)
(d) D ($-\frac{5}{2}$, 0)
(e) E ($^-$6, $\frac{11}{2}$)
(f) F ($-\frac{9}{4}$, $-\frac{9}{2}$)
(g) G (0, $\frac{13}{4}$)
(h) H ($\frac{1}{2}$, $^-$2)
(i) I ($^-$6, $-\frac{3}{2}$)
(j) J (0,0)

8. **Plot all the following points on one coordinate plane:**

(a) A (1.4, 5.3)
(b) B ($^-$5.5, $^-$3.5)
(c) C (6, $^-$2.7)
(d) D ($^-$3, 3.3)
(e) E (0, $^-$1.5)

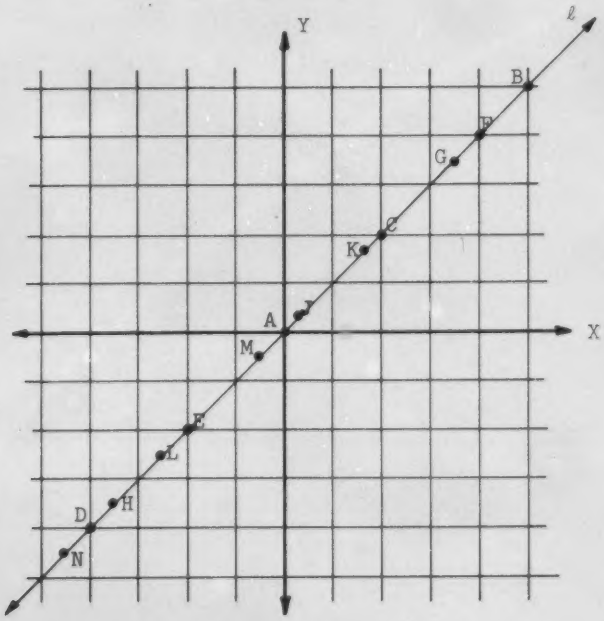
1-12. Graphs in the Plane

The study of geometry with the use of coordinates is called coordinate geometry. This branch of mathematics was started by the mathematician René Descartes in 1637. This invention was a great step forward in mathematics and made possible the discovery of calculus which followed shortly after. In this section, you will learn some very simple coordinate geometry.

Exercises 1-12a

(Class Discussion)

On the coordinate plane below, we have drawn the line ℓ through the point A (0,0) and the point B (5,5).



1. Find the coordinates of some other points on this line. Copy and complete the table below:

Point	A	B	C	D	E	F
X-coordinate	0	5	2	?	?	?
Y-coordinate	0	5	?	?	?	?

2. What do you notice about the coordinates of these points?

3. Now try some points whose coordinates are not integers (the set of integers consists of the number zero, the counting numbers, and the negatives of the counting numbers). Estimate the coordinates as closely as you can.

Point	G	H	J	K	L	M	N
X-coordinate	$3\frac{1}{2}$	?	?	?	?	?	?
Y-coordinate	$3\frac{1}{2}$	?	?	?	?	?	?

4. Does the same relation between the X- and Y-coordinates that you noticed before still seem to hold?

5. What do you conclude about the coordinates of points on this line?

You should have noticed that for all the points that you checked in the last example, the X- and Y-coordinates are equal. This relation holds true for all points on this line. Do you suppose that all points with X- and Y-coordinates equal will lie on this line? Try to find out. Plot these points on the above graph.

$$P\,(1,1),\quad Q(^{-}3,^{-}3),\quad R\,(4.3,4.3),\quad S\,(^{-}2\tfrac{2}{3},^{-}2\tfrac{2}{3}).$$

Do these points seem to lie on the line ℓ? It is true that all points whose X- and Y-coordinates are equal lie on the line ℓ.

Here is another way of looking at what you have just developed. Let (x,y) be a point somewhere in the plane. If $y = x$, then this point lies on line ℓ. If $y \neq x$, then this point does not lie on ℓ. We can think of the line ℓ as the "picture" of the equation $y = x$. We say that ℓ is the graph of the equation $y = x$.

Exercises 1-12b

1. Plot points $(^-1,^-1)$ and $(6,6)$, and draw line ℓ through them.

   (a) Is the point $(2\frac{1}{2}, 2\frac{1}{2})$ on ℓ?

   (b) Is the point $(^-100,^-100)$ on ℓ?

   (c) Is the point $(7, 7\frac{1}{2})$ on ℓ?

   (d) Is the point $(^-5,5)$ on ℓ?

   (e) If (a,b) is on ℓ, what must be true about the numbers a and b?

   (f) If (c,d) is <u>not</u> on ℓ, what must be true about the numbers c and d?

   (g) If x is <u>any</u> number, is the point (x,x) on ℓ?

   (h) Can you find a number n for which the point $(n, n+1)$ is on ℓ?

2. Think of points whose coordinates are $(5,y)$.

   (a) Plot $(5,y)$ where y is 1.

   (b) Plot $(5,y)$ where y is $^-2$.

   (c) Plot $(5,y)$ where y is $\frac{5}{3}$.

   (d) Plot $(5,y)$ where y is $^-4$.

   (e) Plot $(5,y)$ where y is $4\frac{1}{2}$.

   (f) Describe where these points lie.

   (g) The graph of all points whose coordinates are $(5,y)$ is _____.

3. Think of points whose coordinates are $(x,4)$.

   (a) Plot $(x,4)$ where x is 2.

   (b) Plot $(x,4)$ where x is $3\frac{1}{3}$.

   (c) Plot $(x,4)$ where x is $^-3$.

   (d) Plot $(x,4)$ where x is 0.

   (e) Plot $(x,4)$ where x is $^-5$.

   (f) Describe where these points lie.

   (g) The graph of all points whose coordinates are $(x, 4)$ is _____.

4. If the lines for Exercises 1, 2, and 3 were drawn on the same coordinate plane, what would be the coordinates of the point of intersection for the lines formed in

   (a) Exercises 1 and 2?
   (b) Exercises 1 and 3?
   (c) Exercises 2 and 3?

5. Draw the line through points $G(^{-}5,5)$ and $H(5,^{-}5)$.

   (a) Point J has an X-coordinate of 2, and J is on $\overleftrightarrow{GH}$. What is the Y-coordinate of J?
   (b) Point K on $\overleftrightarrow{GH}$ has a Y-coordinate of $^{-}3$. What is the X-coordinate of K?
   (c) Find the missing coordinates for points L, M, N and P which lie on $\overleftrightarrow{GH}$.

Point	L	M	N	P
X-Coordinate	$^{-}5$	?	117	$^{-}117$
Y-Coordinate		$^{-}5$	?	?

6. Pick two points A and B in Quadrant I for which the X-coordinate is twice the Y-coordinate. Plot those two points, and draw $\overleftrightarrow{AB}$.

   (a) Is the point (3,6) on $\overleftrightarrow{AB}$?
   (b) Is the point (6,3) on $\overleftrightarrow{AB}$?
   (c) Is the point (20,10) on $\overleftrightarrow{AB}$?
   (d) Is the point (2n,n) on $\overleftrightarrow{AB}$ for any positive number n?
   (e) Is the point $(^{-}6,^{-}3)$ on $\overleftrightarrow{AB}$?
   (f) Is the point $(^{-}100,^{-}50)$ on $\overleftrightarrow{AB}$?
   (g) Is the point $(^{-}24{,}690\ ,\ ^{-}12{,}345)$ on $\overleftrightarrow{AB}$?

1-13. Summary

Section 1-1.

The physical objects around us suggest the ideas regarding shape and size which are studied in geometry.

We think of points as definite locations and space as the set of all points. Geometric figures are sets of points and therefore are subsets of space.

Section 1-2.

A line is a set of points suggested by a stretched string. Many lines can pass through a single point.

Property 1. Through any two different points in space there is exactly one line.

Lines are named by identifying two of its points.

If two lines intersect, they intersect in exactly one point called the point of intersection.

Section 1-3.

A plane is a set of points suggested by a flat surface extending indefinitely in all directions.

Property 2. If two points of a line lie in a plane then all of the line lies in the plane.

Property 3. For any two points in space, there are many planes which contain them.

Property 4. Through any three points, not all on the same line, there is exactly one plane.

A plane is named by identifying any three of its points not all on a line.

Section 1-4.

When two lines intersect, their intersection consists of exactly one point.

When the intersection of two lines is empty,

(a) they are called parallel if they lie in the same plane.

(b) they are called skew if they do not lie in the same plane.

Section 1-5.

The intersection of a line and a plane may

(a) be empty, in which case, they are parallel,

(b) consist of a single point,

(c) or be the line itself.

Property 5. If the intersection of two different planes is not empty then the intersection is a line.

When we say that two planes are parallel we mean that their intersection is the empty set.

Section 1-6.

When we say that point P is between points A and B, we mean

(a) A, B, and P are collinear, and

(b) on their line the points are in the order A, P, B or B, P, A.

Definition. The segment $\overline{AB}$ is the set of points consisting of the two points A and B, and all points between A and B.

Section 1-7.

A path in a plane is a set of points which can be represented by making a tracing with a pencil on a sheet of paper without lifting the pencil.

A simple closed path is a path in a plane which starts at a point and returns to that point without crossing itself.

Two points are separated by a path p in a plane if the two points are not on path p and if every path connecting the two points intersects path p.

A simple closed path separates the plane into two parts called the interior and exterior.

A line separates the plane into a pair of opposite half-planes or sides.

A point separates the line into a pair of opposite half-lines.

A plane separates space into two half-spaces.

A ray is a half-line together with its boundary point. The boundary point is called the endpoint of the ray.

Two opposite half-lines, each joined with the common boundary point, form a pair of opposite rays.

Section 1-8.

An angle is the union of two non-collinear rays having the same endpoint.

The interior of $\angle ABC$ is the intersection of the A-side of $\overleftrightarrow{BC}$ and the C-side of $\overleftrightarrow{AB}$.

If point P is in the interior of $\angle ABC$, we say that $\overrightarrow{BP}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Definition. For any three non-collinear points A, B, and C, the union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called a triangle.

Section 1-9.

A point may be located in a plane by a pair of numbers (x,y) using an X-axis and a Y-axis. The order of the pair of numbers is important. The pair $(2,3)$ and the pair $(3,2)$ locate different points.

Section 1-10.

The X-axis is separated into two opposite half-lines by the origin, the point of intersection with the Y-axis. Positive numbers are assigned to all the points of the half-line to the right. Negative numbers are assigned to all points on the opposite half-line to the left as opposites of the positive numbers. Positive and negative numbers are assigned to points on the Y-axis in a similar manner.

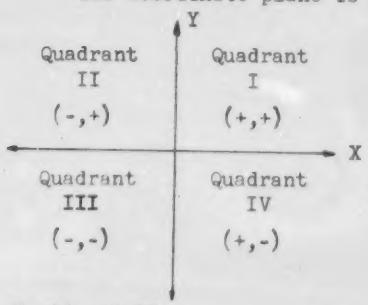
The numbers assigned to the axes are called coordinates.

Section 1-11.

With the assigning of negative numbers to the axes, points may be located in any part of the plane by pairs of numbers (x,y). The number x is called the X-coordinate and the number y is called the Y-coordinate of a point.

The coordinate plane is a plane in which each point is assigned a pair of coordinates and each pair of coordinates designates a point.

The coordinate plane is separated into four quadrants as shown at the left. In each quadrant the X- and Y-coordinates of the points are indicated as positive (+) or negative (-).



Section 1-12.

Points whose X- and Y-coordinates are equal lie on a line ℓ. Then for any point $P(x,y)$, if $y = x$, P is a point on ℓ, and ℓ is the set of all such points. We say that ℓ is the graph of the equation $y = x$.

In this way various lines become graphs consisting of points having X- and Y-coordinates related in certain ways.

Chapter 2

FUNCTIONS

2-1. Car Travel

The relationship between the distance traveled, the speed, and the time, is familiar to everyone who has traveled by car. For example, if you could maintain a speed of exactly 50 miles per hour, you could travel 50 miles in one hour, 100 miles in two hours, and so on. A shorter way of saying this would be:

distance traveled = (miles per hour) × (number of hours).

In terms of the situation mentioned above you could simply write:

$$d = 50 \cdot t,$$

where d represents the number of miles traveled, 50 represents the speed in miles per hour, and t represents the number of hours.

Another way of describing this relationship is to say,

> "The distance traveled depends on the number of hours you have been traveling."

In present day language we would say that,

> "The distance traveled is a function of the time spent in traveling."

The rule, or formula, $d = 50 \cdot t$, says that, if you input a value for t and do the indicated calculation, then you output exactly one corresponding numerical value for d. It is this kind of relationship, called a function, that we want to discuss in this chapter.

Exercises 2-1a

(Class Discussion)

1. Complete the table of values below. For each t we have a distance, $50t$, which corresponds to the input t.

nput	output
t	50t
0	0
1	50
2	?
3	?
4	?
5	250

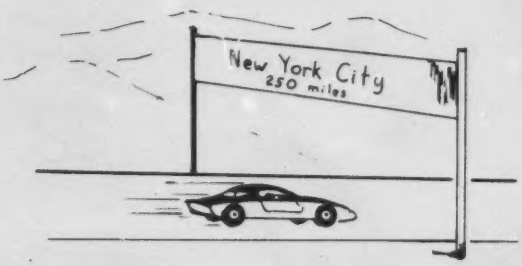


Figure 1

When we use the relationship between d and t, we are actually using a model of the real situation for which a number of necessary details such as heavy traffic, gas stops, and meals seem to be ignored. In the actual planning of an automobile trip, however, we often do estimate our speed and how far we can travel by such a model. You have probably heard people make such remarks as, "You can average 65 miles an hour on that stretch of highway," or "You'll be lucky to average 25 miles an hour on that narrow, winding road." Do these statements mean that the first speaker kept an exact speed of 65 miles per hour and the second speaker was able to drive always at 25 miles an hour? Perhaps, but this is not often true.

It is possible that the trip of 250 miles might have been as follows:

Hour of travel	No. of miles traveled	Comment
1st	65	Interstate Highway had very little traffic.
2nd	50	A different highway had a few traffic lights.
3rd	20	A quick lunch and gas stop was necessary.
4th	60	A return to the Interstate highway followed.
5th	55	The traffic began to get heavy.

That is, if the trip began at noon, by one o'clock the car would have traveled 65 miles; by two o'clock, 50 more miles for a total of 115 miles. The average speed after two hours would be $57\frac{1}{2}$ miles per hour.

Exercises 2-1b

(Class Discussion)

1. Putting the above information in another form, add to this information by completing the following table:

	(m/p/h)	(Total distance)	(Average speed)
Time	No. miles added each hour	No. miles of total travel at end of each hour	No. miles per hour averaged at end of each hour
t = 1	65	d = 65	65
t = 2	50	d = 115	$57\frac{1}{2}$
t = 3	20	d = 135	?
t = 4	60	?	?
t = 5	55	?	50

2. Another example of a functional relationship is the correspondence between the number of the year and a person's age in years. This relationship can be shown on a graph. We show below the graphs of the ages of two brothers during a ten-year period. Bob was exactly 8 years old when John was born in 1950. Just recently, the brothers were wondering about the relationship of their ages at different times. Can you answer their questions from the graphs given below?

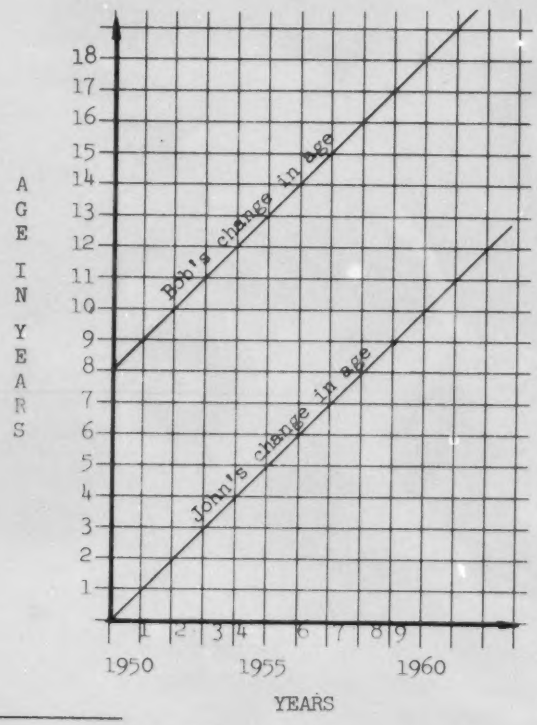
(a) Can Bob and John ever be the same age?

(b) Was Bob ever twice the age of John? If so, when?

(c) Was Bob ever three times the age of John? If so, when?

(d) Was Bob ever five times the age of John? If so, when?

(e) Was Bob ever nine times the age of John? If so, when?



In Exercise 1 above, the correspondence between the "time" and the "average speed" was easily shown in a table. This method of representing a function is very often the only way that such information can be shown.

In Exercise 2 an important reason for studying functional relationships is illustrated. Namely, that by "looking at" functions you can find additional, important information that usually is not apparent in the original form of the given situation. In this situation the representation of the functions by graphs should have made it easier to discover the answers to the questions.

2-2. Falling Objects

Near the end of the sixteenth century (around 1590), as the story goes, Italian mathematician and scientist, Galileo, performed an important experiment in the city of Pisa, which is located on the Arno River in central Italy. He hauled a ten-pound weight and a one-pound weight to the top of the now famous leaning tower of Pisa and dropped them off at the same time. Friends of Galileo on the ground below observed that the weights remained side by side as they fell and that they seemed to hit the ground at exactly the same instant.

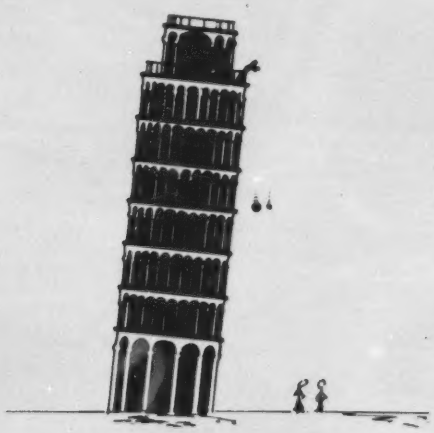


Figure 2

This simple experiment upset many people, for almost 2000 years earlier (about 350 B. C.), the great Aristotle had argued that the heavier the object the faster the fall. During the many years between, no one had really tested Aristotle's statement.

Returning to the experiment in Pisa, let's see what it is that Galileo discovered. Since the two stones were always side by side as they fell, we say that for any given number of seconds the two weights will travel the same distance.

What Galileo discovered was the the distance traveled by a falling object does not depend on its weight, but only depends on the length of time during which it has been falling. It is more common now to refer to falling bodies, rather than falling objects, but the meaning is, of course, the same. In present day language we would say that the distance traveled by a falling body is a function of the time spent in falling.

Galileo wanted to describe this relationship using the language of mathematics; he wanted a mathematical model for what physically happened.

Thus a rule or formula for finding the distance traveled will express the distance in terms of the time and will not involve the weight. This

relationship, which we will not attempt to justify here, turns out to be:

> The number of feet traveled by a falling object is approximately equal to the product of 16 and the square of the number of seconds the object was falling.

Using a new symbol, $\approx$, to mean "approximately equal to", we shorten this statement to:

$$d \approx 16 \cdot t \cdot t,$$

or the more usual notation,

$$d \approx 16 \cdot t^2$$

where d is measured in feet, and t is measured in seconds.

For convenience, we often write this statement as follows:

$$d = 16t^2.$$

We should keep in mind that the answers are not exact in the physical sense, for not only is accurate, split-second timing almost impossible, but the "16" is an approximation that has been rounded off to the nearest whole number. (The meaning of the "16", in the statement, will be discussed in later courses.)

Let us consider an example of how this formula might be used.

Example: Suppose that two boys wished to measure the height of a waterfall, using the formula, $d = 16t^2$, and timing the fall with a stop watch. Actually, physical conditions, such as whether the cliff is vertical or slants outward at the bottom, and whether the spray of the water blocks vision, often make such experiments difficult to perform. Let us assume, however, that the very best conditions exist in this situation.

One boy went to the top of the falls, and the other stayed at the bottom of the falls with a stop watch. The boy on the cliff dropped a rock to the ground below. The other boy started the stop watch when the stone was dropped and stopped it when the stone hit the ground. He found that 3.5 seconds had passed. Now the boys made their calculations. They started with the formula that tells them that the height (in feet) of the falls can be estimated by the product of 16 and the square of the time (in seconds). This is how they made their calculations.

In this formula

$$d \approx 16t^2,$$

they replaced t by 3.5 to obtain

$$\begin{aligned} d &\approx 16(3.5)^2 \\ &\approx 16(3.5)(3.5) \\ &\approx 16(12.25) \\ &\approx 196 \end{aligned}$$

So the boys concluded that the falls were almost 200 feet high. Of course, it would be very difficult for these boys to be exactly precise in using the stop watch. Although a stop watch measures time to the tenth of a second, for this experiment a mistake of a fraction of a second in measuring the time would produce quite different results, since the stone drops 112 feet between the third and fourth seconds!

Figure 3

In the following exercises you will find out a little more information about the distance traveled by the rock used in measuring the approximate height of the falls mentioned in the example above.

Exercises 2-2

1. Using the formula $d = 16t^2$, find the approximate distance traveled by the falling stone during each half-second during its fall. Fill in the missing values in the following table:

t : number of seconds

If t =	0	0.5	1	1.5	2	2.5	3	3.5
then d =	0	4	?	36	?	100	?	196

d : number of feet rock has fallen

2. Now find out how far the rock falls each half-second. Using the information of the table in Exercise 1, complete the following table.

During the following half-second intervals	Number of feet rock fell
t = 0 to t = .5	4
t = .5 to t = 1	?
t = 1 to t = 1.5	?
t = 1.5 to t = 2.0	?
t = 2.0 to t = 2.5	?
t = 2.5 to t = 3.0	?
t = 3.0 to t = 3.5	?

3. Suppose the cliff had been somewhat higher; from your table in Exercise 2, can you conjecture (this means "make an intelligent guess") how many feet the rock would drop between $t = 3.5$ and $t = 4$? between $t = 4$ and $t = 4.5$? between $t = 4.5$ and $t = 5.0$? (Do not use the formula $d = 16t^2$. This problem needs very little arithmetic.)

In using the rule, or formula,

$$d = 16t^2,$$

we substitute or "input" a number as a value for t and then calculate or "output" the numerical value for d. For each input there is a corresponding output. This correspondence between the inputs and the outputs is an example of a "function". The formula $d = 16t^2$ serves to specify the function, but the formula itself is not considered to be the function. Thus

$$y = 16x^2$$

can be considered as designating the same function as $d = 16t^2$ does, because it produces the same correspondence between input and output values.

During this discussion you may have doubted that a feather would really fall as fast as a stone. Galileo was well aware that the feather falls more slowly, and he assumed that the air resistance was very great with such a light object. Later experiments showed this to be true. When a feather and a lead ball are dropped in a closed container from which all the air

has been pumped they fall side by side and strike the bottom at the same time. The answers obtained using the formula $d = 16t^2$ are regarded as the correct answers in terms of the model, and are often good estimates of the real physical problem. For solid objects, such as lead balls, falling through short distances, the model is in rather close agreement with real life.

2-3. Some Examples of Functions

Functions occur everywhere in mathematics as well as in situations which may not be strictly mathematical. Because they occur so often we shall have many occasions to discuss them. We cannot hope to give a list of examples which will suggest all the possibilities. However, we shall give a few examples to suggest the wide variety of places in which the function idea is used. Let's begin by summarizing the information from the last section.

In the last section it was mentioned that the distance traveled by a falling body is a function of the time spent in falling. Here, it was noted that by substituting or "inputting" values for t (in seconds), we can calculate or "output" values for d (in feet). For each input, there is a corresponding output, or image. This correspondence between the acceptable inputs and the outputs is called a function. The mathematical sentence, $d = 16t^2$, helps to describe how the correspondence is to be made, and is only part of the description of the function. Other parts of the description of the function include the set of inputs and the set of outputs.

It sometimes helps our understanding to think of a function as a kind of machine (see Figure 4). This machine is equipped with a hopper and a spigot. When we place an input into the hopper the machine produces the corresponding output at the spigot.

Thus, for our falling body function, the output, or image, will always be 16 times the square of the input.

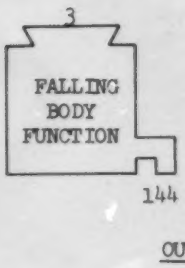


Figure 4

In this example, we input a number of seconds and output a distance that corresponds to the number of seconds. Thus, with the inputs 1, 2, 3, 4, and 5, we get the outputs 16, 64, 144, 256, and 400. This can be shown as follows:

$1 \longrightarrow 16$, $2 \longrightarrow 64$, $3 \longrightarrow 144$,

$4 \longrightarrow 256$, $5 \longrightarrow 400$, and so on.

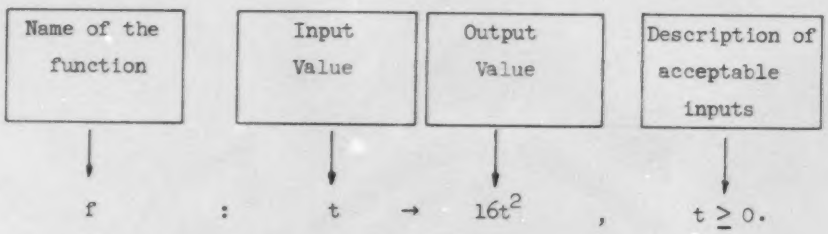
In general, an input of t (seconds) outputs $16t^2$ (feet), and diagrammatically, this is indicated in a fashion similar to the above:

$$f : t \rightarrow 16t^2.$$

The letter f is a name for the function. The remainder of the notation states that under this function, $16t^2$ corresponds to t. In this case, f names a function whose input is a number and whose output is a number. Since the values of t (time) are never negative in this function, the condition $t \geq 0$ (t is greater than or is equal to 0) can be included in the description. Thus, we can now write:

$$f : t \rightarrow 16t^2, \; t \geq 0.$$

Schematically, we can think of this statement in the following way:



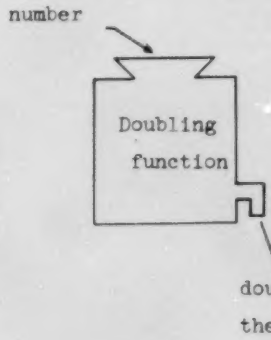
This way of representing a function is called arrow notation, and it can be read as follows:

(1) "the function f such that to every t there corresponds $16t^2$, and t is a non-negative number", or

(2) "the function f such that t is the input and $16t^2$ is the output, and t is a non-negative number", or simply

(3) "f takes t into $16t^2$, where $t \geq 0$".

Two other examples of functions, which can be illustrated by a machine, are the doubling function and the squaring function.



For example:

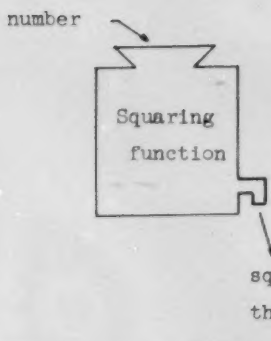
1 has the image 2, or $1 \rightarrow 2$.

2 has the image 4, or $2 \rightarrow 4$.

3 has the image 6, or $3 \rightarrow 6$.

4 has the image 8, or $4 \rightarrow 8$.

The arrow notation for this function is $f : n \rightarrow 2n$.



For example:

1 has the image 1, or $1 \rightarrow 1$.

2 has the image 4, or $2 \rightarrow 4$.

3 has the image 9, or $3 \rightarrow 9$.

4 has the image 16, or $4 \rightarrow 16$.

The arrow notation for this function is $f : n \rightarrow n^2$.

Suppose you say, "The time I wake up in the morning is a function of the time I go to bed, and I always sleep 7 hours." If t represents the time you go to bed, then $t + 7$ is the time you wake up. We can describe this function in arrow notation by

$$f : t \rightarrow t + 7.$$

The f, used here, is probably the most common name for a function, but F, g, h, and other letters are also used as names for functions. We could

just as well describe this function by using these other letters, for example,

$$F : t \rightarrow t + 7$$

$$g : t \rightarrow t + 7$$

$$h : t \rightarrow t + 7$$

Functions may establish correspondences between objects other than numbers.

Example 1:

Names of Four Buildings in New York City	Height (in feet)
Empire State Building	1472
Chrysler Building	1046
RCA Rockefeller Center	850
Pan Am Building	808

We have four inputs and the corresponding output for each. In this case, the height (output) is a function of the building (input).

Example 2:

The world's tallest known trees are all in California. If we input the name of one of these big redwood giants, we output the height; for example, the height is a function of the name of the tree.

Names of Six Redwood Trees	Height (in feet)
Howard Libbey Tree	367.6
Harry W. Cole Tree	365.4
National Geographic Tree	364.5
Rockefeller Tree	356.5
Founders Tree	352.6
Redwood Creek Grove	352.3

It is also true that if we input the heights of these particular trees we would output the name of the tree. In this case, the name of the tree is a function of the height.

It is important to note that when we reverse the direction of a correspondence which is a function, the resulting correspondence is not always a function.

Example 1:

Below is a table which shows a correspondence between pairs of numbers, a correspondence from the top row of numbers to the bottom row.

Input	0	1	2	3	4	5	6
Output	$12\frac{1}{2}$	8	$4\frac{1}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$

You can see that corresponding to each input there is exactly one output. Therefore, this correspondence is a function.

If the direction of the above correspondence is reversed, then the table below shows the following correspondence between pairs of numbers.

Input	$12\frac{1}{2}$	8	$4\frac{1}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$
Output	0	1	2	3	4	5	6

You can see that there are two outputs, 4 and 6, corresponding to the input $\frac{1}{2}$. Therefore, this second correspondence is not a function because there is more than one output for a particular input.

Example 2:

Every article in your neighborhood hardware store has a price. The price is a kind of label associated with the article. Thus, the price is a function of the article.

We may describe the correspondence by such a statement as, "Every article in this store has a price." Or, we might say, "Associate with every article in the store its price." Since, for each article there is one price, we have a functional relationship. We can use the arrow notation to describe this function, where price is a function of the article,

$$p : \text{article} \rightarrow \text{price}.$$

Now, suppose we say, "Associate with every price (for which the store sells goods) an article." Since, for some prices, there would be not one, but several articles, we do not have a functional relationship.

Exercises 2-3a

(Class Discussion)

1. In some cases below, the statement describes a function. In other cases, the statement does not describe a function. If the statement describes a function use the arrow notation to represent the function. If the statement does not describe a function explain why it does not.

   (a) Associate with a point in space a line through the point.

   (b) Associate with two different points in space a line through these points.

   (c) Associate with a point in space a plane through the point.

   (d) Associate with two different points in space a plane through these points.

   (e) Associate with three non-collinear points in space a plane through these points.

Exercises 2-3b

Below are four statements which describe correspondences between pairs of things. Write each statement on a sheet of paper, and, under each of them, write a new statement which shows that the first correspondence has been reversed. You should have 8 statements (4 pairs) on your paper.

Now, follow these instructions:

(1) Decide whether the first statement, in each pair of statements, describes a function, and then

(2) decide whether the new statement, in each pair of statements, describes a function.

(3) If any of these 8 statements describe a function, then represent this function in terms of arrow notation.

1. (a) Associate with each member of your mathematics class his or her age in years.

   (b) Associate with each state of the United States the name of its present governor.

(c) Associate with each height, in inches, a citizen of the United States.

(d) Associate with a first name for a person, a last name.

2-4. Ways of Representing Functions

We have seen that functions may be represented by using the arrow notation. Since a function is a correspondence it is reasonable to expect that a function may be represented in many other ways. In this section, we will discuss some of the more common ways of expressing functions.

(a) A function may be represented by a statement. For example:

With two distinct points in the plane associate the distance between them.

This may be expressed by using the arrow notation:

$$f : (A,B) \rightarrow \text{distance between } A \text{ and } B,\ A \neq B.$$

Here, inputs are pairs of points in the plane, while outputs are positive numbers.

(b) A function may be expressed by a table. For example:

In a basketball game, the number of points scored by each player on the Lincoln High School team is shown in the following table.

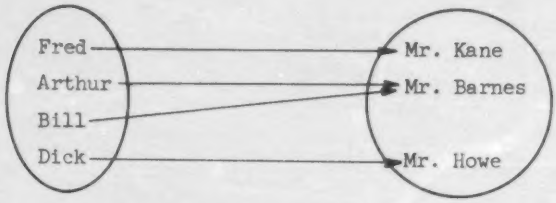
Player	Points
Andrews	12
Brown	8
Davis	9
Taylor	10
Harris	6
Faber	10
Elliot	7

In this case, we have a function from a set of players to a set of positive numbers. The table actually states the correspondence completely and is probably the most useful form of representing the function. However, we may indicate the function by using the

arrow notation, as follows:

h : player → number of points scored by player.

(c) A function may be indicated by a diagram which pictures the correspondence. For example:



This diagram shows the correspondence between a set of boys and the set of their fathers. Although we could do so, it would be of little use to express this function by using the arrow notation.

(d) A function may be described by a formula. Think of a formula as a rule which has been abbreviated and written in the form of a mathematical sentence. For example:

$$P = 4s.$$

In this formula, s (representing inputs) stands for the length of a side of a square and P (representing outputs) stands for the perimeter of the square (see Figure 5).

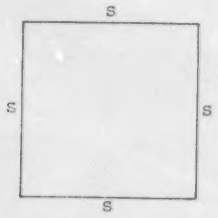


Figure 5

In describing a function by a formula, such as $P = 4s$, the letter representing the output occurs only once in the formula and is alone on one side of the equality sign. If this function

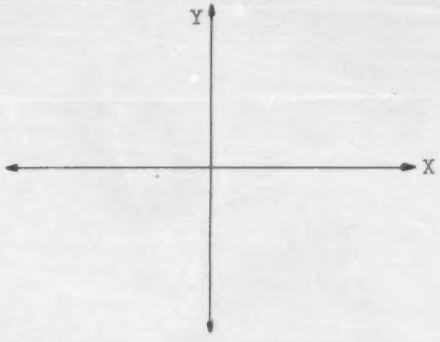
is named g, then it can be represented in arrow notation as:

$$g : s \to 4s,\ s > 0.$$

Remember: $s > 0$ means that "s is greater than 0".

(e) A graph is one of the most useful ways of representing a function. Before we learn to draw the graph of a function, let's agree to the following statements:

(1) "To draw the graph of a function" means to find a collection of points in the coordinate plane



such that they will accurately picture the certain kind of association or correspondence which we call a function.

(2) Inputs of a function will be associated with the coordinates of points on the X-axis (the horizontal number line).

(3) Outputs of a function will be associated with the coordinates of points on the Y-axis (the vertical number line).

Now, we will draw the graph of the function, h, described by

$$h : x \to 2x,\ x \geq 0.$$

Remember: $x \geq 0$ means that "x is greater than or is equal to 0".

In order to draw the graph we will first choose a few inputs, represented by x, and calculate the corresponding outputs, represented by 2x. Our results are shown below.

Input: x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
Output: 2x	0	1	2	3	4	5	6	7	8

Notice that inputs were chosen so that the calculation of outputs was easy work!

Notice also that an increase in the input causes an increase in the output.

Now, we plot the points corresponding to these pairs of numbers: (0,0), $(\frac{1}{2}, 1)$, (1,2), $(\frac{3}{2}, 3)$, (2,4), $(\frac{5}{2}, 5)$, (3,6), $(\frac{7}{2}, 7)$, (4,8).

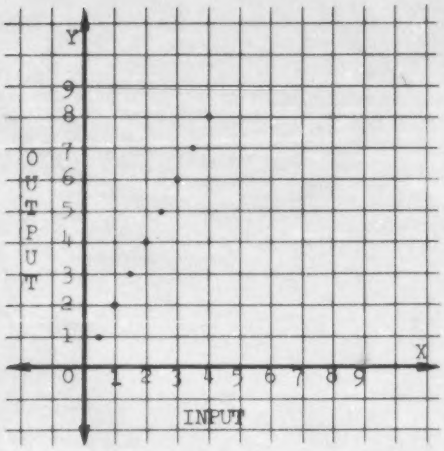


Figure 6a

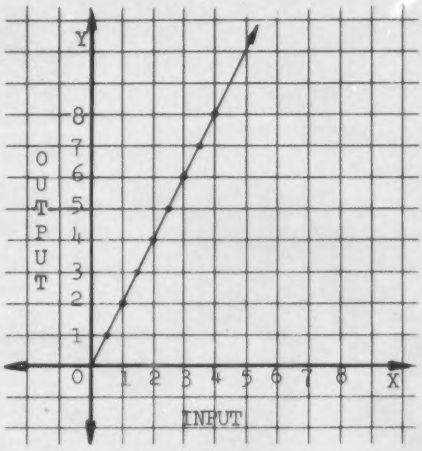


Figure 6b

Figure 6a shows these points. Of course, the graph of this function consists of more points than are shown. You can see that it would not be possible to make a complete table for the function. Therefore, it would also be impossible to plot all the points belonging to its graph. However, as shown in Figure 6b, the points we have plotted seem to be collinear. In fact, if we find other number pairs which belong to this function, such as $(\frac{1}{4}, \frac{1}{2})$, $(\frac{3}{2}, 9)$, (5,10), their corresponding points seem to be collinear with those already plotted. Let's agree that the graph of this function is a ray, whose endpoint is the origin, and which passes through the first quadrant as shown in Figure 6b. Notice that this graph is rising from left to right. This should not surprise us because we have already noted that an increase in the input causes an increase in the output. A function whose graph has this characteristic is often called an increasing function.

To gain more experience in drawing graphs of functions, we will draw the graph of the function, f, described by

$$f : x \to x^2,\ 0 \le x \le 3.$$

Remember: $0 \le x \le 3$ means that "$x \ge 0$ and $x \le 3$".

Again, in order to draw the graph we will first choose a few inputs, represented by x, and calculate the corresponding outputs, represented by x^2. Our results are shown below.

Input: x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
Output: x^2	0	$\frac{1}{4}$	1	$\frac{9}{4}$	4	$\frac{25}{4}$	9

Again, notice 2 things:

(1) inputs were chosen so that the calculation of outputs was easy work;

(2) an increase in the input causes an increase in the output.

Now, we will plot the points corresponding to these number pairs: (0,0), $(\frac{1}{2}, \frac{1}{4})$, (1,1), $(\frac{3}{2}, \frac{9}{4})$, (2,4), $(\frac{5}{2}, \frac{25}{4})$, (3,9).

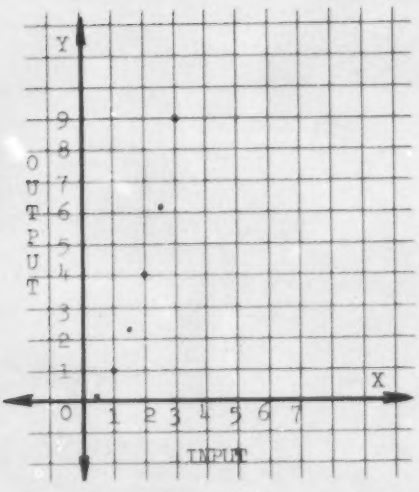


Figure 7a

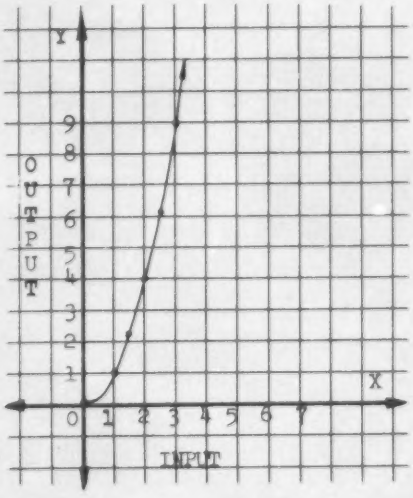


Figure 7b

Figure 7a shows these points. Of course, the graph of this function also consists of more points than are shown. You can see that it would not be possible to make a complete table for this function either. Therefore, it would, again, also be impossible to plot all the points belonging to its graph. However, it is easily seen that the points we have plotted are not collinear. But, by drawing a smooth curve through the plotted points, as shown in Figure 7b, we can obtain a fairly accurate sketch of the true graph of the function. Notice that this graph, too, is rising from left to right. Recall that an increase in the input causes an increase in the output. Therefore, we should expect the graph of this function to behave as it does (rise from left to right). Thus, this function, too, can be called an increasing function.

When drawing the graph of a function, we may choose any unit of measure for the X-axis and any unit of measure for the Y-axis. This information will be useful when drawing the graphs of certain functions.

We have learned to draw the graph of a function by making a table of inputs and outputs. Now, let's learn to read the graph of a function (see Figure 8).

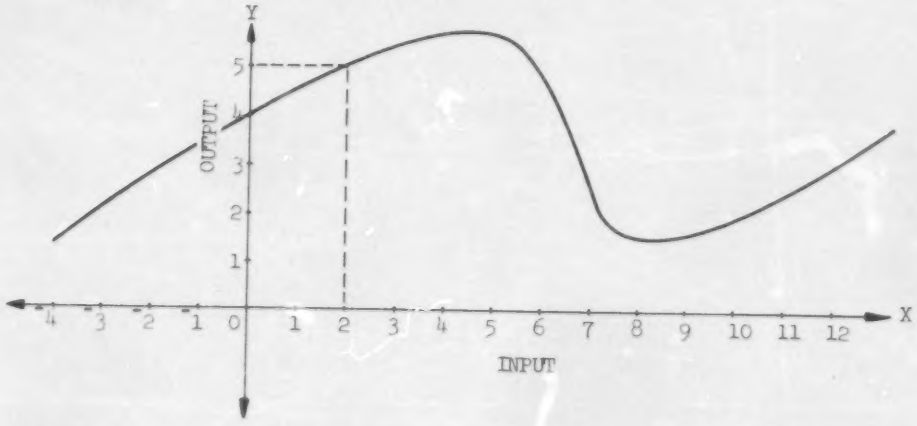


Figure 8

We can read this graph in the following way. Suppose we wish to know the output for the input 2. We first locate the point labeled 2 on the input axis (here the horizontal axis). Then we search along the vertical line through this point until we find a point on the graph. The horizontal line through this point of the graph passes through 5 on the output axis. Hence, the point with coordinates (2,5) is a point on the graph, and 5 is the output corresponding to the input 2. We can see that by using this method, a single output can be found for each input. The reason is that each vertical line intersects the graph in just one point.

It would be difficult to express this function by using the arrow notation. In some cases, a graph is the only convenient way of representing a function.

In Figures 9a and 9b, we see graphs which do not represent functions.

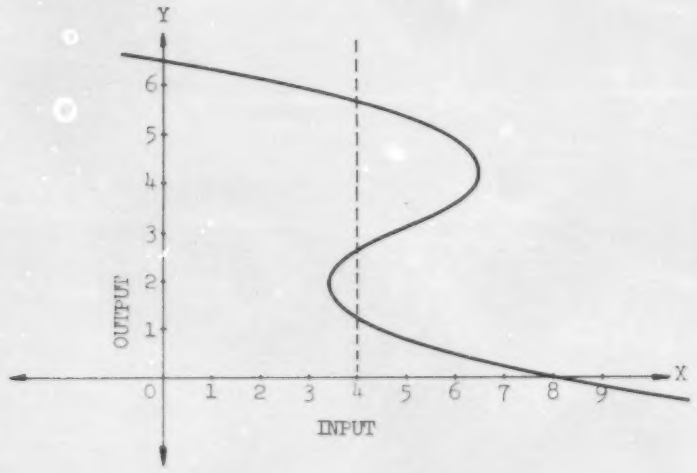


Figure 9a

In Figure 9a, what would be the output corresponding to an input of 4? Here, we see that instead of getting a single definite image, three possible images are put forth. Thus, there is no one output corresponding to each input. So, although this graph may be useful for other purposes, it does not represent a function.

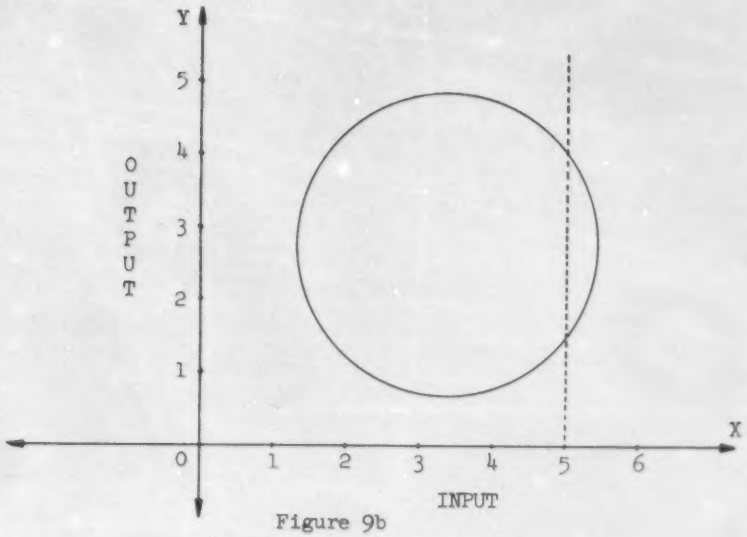


Figure 9b

In Figure 9b, what would be the output corresponding to an input of 5? Here, we see that this input will be associated with two images. Again, there is no one output corresponding to each input. Therefore, this graph does not represent a function either.

Check Your Reading

1. Can you name 5 ways of representing a function?
2. What does "to draw the graph of a function" mean?
3. Which axis in the coordinate plane is associated with the set of inputs of a function?
4. Which axis in the coordinate plane is associated with the set of outputs of a function?
5. What is a function called if its graph rises from left to right?
6. What does "to read the graph of a function" mean?
7. When you look at a graph, how can you decide whether it represents a function?

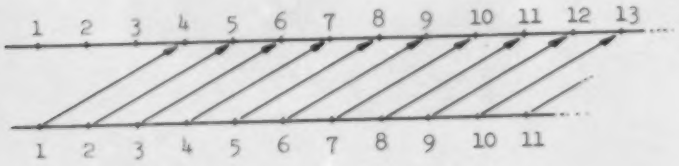
Exercises 2-4

1. Associate with each counting number, c, its triple, 3c.

   (a) Express this function by using the arrow notation.

   (b) Is it practical to express this function by means of a table? Give a reason for your answer.

2. The following diagram represents a function.



   (a) What is the input if the output is 6?

   (b) What is the output if the input is 2?

   (c) What is the output if the input is 10?

   (d) Express this function by using the arrow notation.

3. Study the arrow notation describing the function n.
   $n : w \rightarrow 2w + 1$, w represents a whole number.

   (a) What is the output for an input of 4?

   (b) What is the input for an output of 15?

   (c) Can an output be an even number? Give a reason for your answer.

4. A car travels at a steady rate of 40 miles per hour. Associate with the time (t) in hours the distance (d) in miles covered by the car.

   (a) Express this function by a formula.

   (b) Express this function by using the arrow notation.

   (c) What is the output if the input is $3\frac{1}{2}$?

   (d) What is the input if the output is 150?

5. The function, g, associates with each positive number s the area of a square whose side is s units in length.

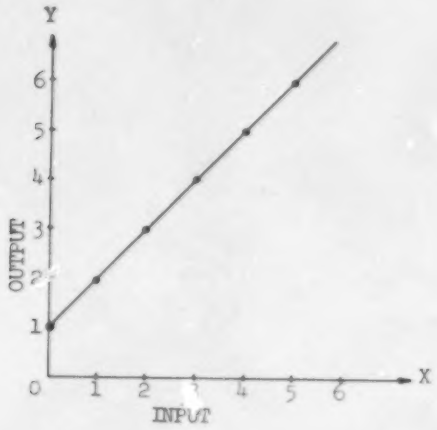
$$g : s \rightarrow s^2, \; s > 0.$$

(a) Find the outputs for inputs of 3 and 6. How are these outputs related?

(b) Find the outputs for inputs of 4 and 8. How are these outputs related?

(c) Does the same relationship between inputs and outputs hold for <u>any pair</u> of inputs such that one is double the other?

6. The graph below represents a function.



(a) What is the output for an input of 2?

(b) What is the output for an input of 5?

(c) What is the input for an output of 4?

(d) Does this graph represent an <u>increasing function</u>? Explain.

7. Draw graphs of the following functions.

(a) $k : x \to 2$.

(b) $\ell : x \to -1$.

Use the language of geometry to accurately describe these graphs.

Explain why functions such as k and ℓ are often called <u>constant</u> functions.

8. In measuring temperature we ordinarily use the Fahrenheit scale. However, in many parts of the world and also in the science laboratory, the Celsius scale (formerly called Centigrade scale) is used. The

relationship between the two scales is a function. The function may be expressed by the following formula.

$$F = \frac{9}{5} C + 32$$

FAHRENHEIT THERMOMETER

CELSIUS THERMOMETER

(a) What is the output for an input (C) of 0? What is the scientific significance of this output?

(b) What is the output for an input (C) of 100? What is the scientific significance of this output?

(c) Complete the given table:

Input: C	0	10	40	60	80	100
Output: $\frac{9}{5} C + 32$	?	?	?	?	?	?

(d) Choose convenient units of measure for both axes, and draw the graph of this function.

9. A certain amount of a gas shut up in a container exerts pressure on the walls. If the size of the container can be changed, as by a piston, then the volume (V) of the container determines the pressure, in agreement with the function

$$g : V \to \frac{50}{V}, \quad V > 0.$$

(a) Complete the given table:

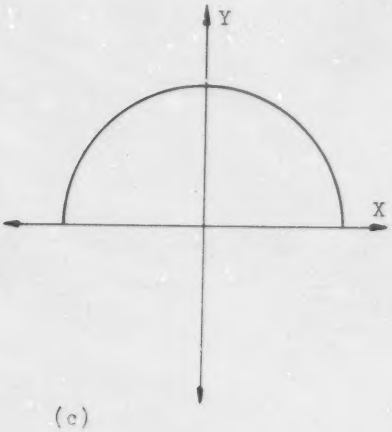
Input: V	1	2	4	5	10	20	25	50
Output: $\frac{50}{V}$	?	?	?	?	?	?	?	?

(b) Draw the graph of this function.

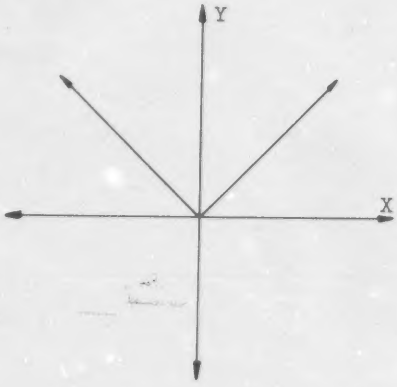
(c) Is the function, g, an increasing function? Explain.

10. Which of the following graphs represent functions?

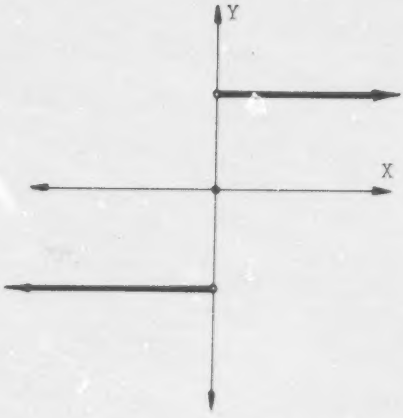
(a)



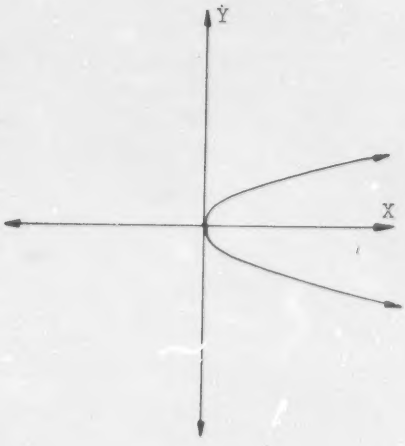
(b)



(c)



(d)



2-5. Discovering the Usefulness of Graphs

We have seen that graphs can be used to give clear pictures of different kinds of functions. More than just a picture of some relationship, a graph can show other important patterns of behavior of a function on closer examination. Consider the following example of charting growth in height.

Example: Tom Jones' dad liked to keep records of facts and figures. When Tom was born and on each of his successive birthdays, Mr. Jones wrote down Tom's height in inches. He stopped keeping records when Tom was 21 years old. Here are his records.

Age	0	1	2	3	4	5	6	7
Height	21	$25\frac{1}{4}$	$28\frac{3}{4}$	$31\frac{1}{4}$	$33\frac{1}{2}$	$35\frac{1}{2}$	$37\frac{1}{4}$	$39\frac{1}{4}$

Age	8	9	10	11	12	13	14
Height	41	44	$46\frac{1}{2}$	$48\frac{3}{4}$	51	$52\frac{3}{4}$	$54\frac{1}{4}$

Age	15	16	17	18	19	20	21
Height	56	$58\frac{1}{4}$	64	71	71	71	71

In some of our other examples, the tables don't completely represent the function. If the number of acceptable inputs is not too many, then by showing all these inputs together with their corresponding outputs, a complete representation of the function can be given. Thus, the information that we have above, in table form, on Tom's height at each birthday through his twenty-first, represents the entire function: namely, Tom's height-at-birthday function. If we graph the function, Tom's growth pattern shows up more clearly.

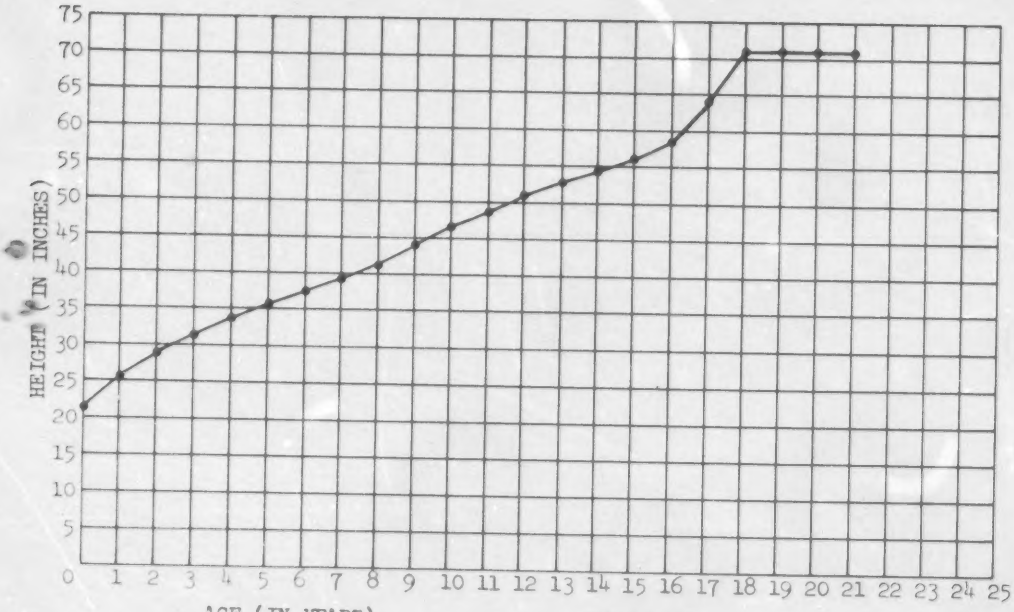


Figure 10

There are two things to notice about this graph. First, we have used different scales on the two axes for convenience in representing the facts. Second, although only the points plotted belong to the graph of the function, we have joined consecutive points by segments to make the graph easier to read.

We cannot read the graph as well as we can read the table. However, we can find a number of properties of the height-at-birthday function from the graph more quickly than we can from the table. We will discuss a number of these properties.

Before doing so, let us agree to use the symbol h (suggesting height) as the name for this function. Throughout this section, h represents Tom's height-at-birthday function:

h : age → height at birthday for the age.

Exercises 2-5a

(Class Discussion)

Look at Figure 10.

1. (a) Is the graph of the function h rising or falling, from left to right, for the inputs (ages) from 0 to 18? Explain.

   (b) For these inputs, is function h an increasing or a decreasing function?

   (c) What meaning can we give to the answer to question 1(a)?

2. (a) What is happening to the graph of the function h for the inputs (ages) from 18 to 21? Explain.

   (b) What meaning can we give to the answer to question 2(a)?

   (c) What word do you think we can use to describe the behavior of h for the inputs (ages) from 18 to 21?

3. The graph of the function h consists of 22 points. Perhaps we can discover some new information by examining the segments joining any two consecutive points belonging to this graph.

   (a) Is there a segment which seems to be steeper than any other segment?

   (b) What meaning can we give to the answer to question 3(a) if we want to relate it to Tom's growth pattern?

   (c) Can we find the coordinates for both of the endpoints belonging to this "steepest" segment by looking at the table of Tom's age and height?

   (d) What are the coordinates of the left endpoint and the right endpoint of this segment?

   (e) Between which two birthdays did Tom have his greatest growth?

   (f) What arithmetic problem can we do to find Tom's growth between these two birthdays? How much did Tom grow during this 1 year period?

   (g) Can we tell from looking at the graph of h how Tom's growth between his 16th and 17th birthdays compares with any other 1 year period?

(h) What arithmetic problem can we do to find Tom's growth between his 16th and 17th birthdays? How much did Tom grow between these birthdays?

The class discussion exercises suggest that we consider how much Tom grew from one birthday to the next. The change in his height during each year may be associated with his age at the end of the year. In this way we describe a new function. In Exercise 3(f) we found that Tom grew 7 inches between his 17th and 18th birthdays; thus his growth during his 18th year was 7 inches. We make 7 correspond to 18 under the "growth function". If we represent the growth function by g (suggested growth), we have

g : age → growth between preceding birthday and birthday for the age.

The growth is, of course, the change in height. Thus the output under g is given by

(height at birthday of given age) minus (height at preceding birthday).

Also, in Exercise 3(h), we found that when 17 is an input for the growth function g, then the output is $5\frac{3}{4}$.

The graph of the function h (see Figure 10) rises rather steadily between Tom's second birthday and his sixteenth birthday. This represents the fact that Tom grew at a fairly steady rate during this period. In terms of the growth function g, this statement means that the outputs for g are nearly the same for all these inputs. Let us check on this by studying the pattern of Tom's growth in more detail. We may do this by going to the table for the function h and using the facts given there to find outputs under the growth function.

We now give <u>part</u> of the table for the growth function g. Then we will use this table to answer different questions about Tom's growth.

Input for growth function g	Growth from age	to age	Output for growth function g
3	2	3	$31\frac{1}{4} - 28\frac{3}{4} = 30\frac{5}{4} - 28\frac{3}{4} = 2\frac{2}{4} = 2\frac{1}{2}$
4	3	4	$33\frac{1}{2} - 31\frac{1}{4} = 33\frac{2}{4} - 31\frac{1}{4} = 2\frac{1}{4}$
5	4	5	$35\frac{1}{2} - 33\frac{1}{2} = 2$
6	5	6	$37\frac{1}{4} - 35\frac{1}{2} = 36\frac{5}{4} - 35\frac{2}{4} = 1\frac{3}{4}$
7	6	7	$39\frac{1}{4} - 37\frac{1}{4} = 2$
8	7	8	$41 - 39\frac{1}{4} = 40\frac{4}{4} - 39\frac{1}{4} = 1\frac{3}{4}$
9	8	9	$44 - 41 = 3$
10	9	10	$46\frac{1}{2} - 44 = 2\frac{1}{2}$
11	10	11	$48\frac{3}{4} - 46\frac{1}{2} = 48\frac{3}{4} - 46\frac{2}{4} = 2\frac{1}{4}$
12	11	12	$51 - 48\frac{3}{4} = 50\frac{4}{4} - 48\frac{3}{4} = 2\frac{1}{4}$

(1) During the ten-year period from age 2 to age 12, what was Tom's largest yearly growth? When did this occur? By examination of the outputs shown in the table we find the largest output is 3. So Tom's greatest yearly growth during the ten-year period was 3 inches, and this happened between the ages of 8 and 9.

(2) During the ten-year period, what was Tom's smallest yearly growth? When did this occur? The smallest output shown in the table is $1\frac{3}{4}$. Do you find that there were two years during which he made the least growth? The first was between his fifth and sixth birthdays, and the second was between his seventh and eighth birthdays.

Now consider another question concerning the ten-year period between Tom's second and twelfth birthdays. What was Tom's <u>average rate of growth</u> during this time? Another way of putting this question is: "Suppose that Tom had grown the <u>same amount</u> each year from the age of 2 to the age of 12. What would this amount have had to be so as to produce the same total growth over the ten-year period?" We can answer this question by finding the change in Tom's height between the ages of 2 and 12, and dividing this total growth into 10 equal parts.

Height at age 12 was 51 inches.

Height at age 2 was $28\frac{3}{4}$ inches.

Total growth from age 2 to age 12 was $22\frac{1}{4}$ inches

because $51 - 28\frac{3}{4} = 50\frac{4}{4} - 28\frac{3}{4} = 22\frac{1}{4}$.

$$\text{Average growth per (for each) year} = \frac{\text{Total growth}}{\text{Number of years}}$$

$$= \frac{22\frac{1}{4}}{10}.$$

We can simplify this fraction by multiplying by the number 1 (named as $\frac{4}{4}$):

$$\frac{22\frac{1}{4}}{10} = \frac{4}{4} \cdot \frac{22\frac{1}{4}}{10} = \frac{4 \cdot 22\frac{1}{4}}{4 \cdot 10} = \frac{88 + 1}{40} = \frac{89}{40} = 2\frac{9}{40}.$$

If we remember that the numerals "$\frac{1}{4}$" and ".25" are just different names for the same number, then we could simplify this fraction by replacing "$22\frac{1}{4}$" by "22.25":

$$\frac{22\frac{1}{4}}{10} = \frac{22.25}{10} = 2.225.$$

In any case, we see the <u>average rate of growth</u> is just about $2\frac{1}{4}$ inches. <u>Remember</u>: $2\frac{9}{40} = 2.225$ and <u>either</u> numeral names a number a little less than $2\frac{1}{4}$.

Exercises 2-5b

For these exercises, refer to the table and graph of Tom's height-at-birthday function h (see pages 81-82).

1. We know from the class discussion exercises that Tom grew more between his 17th and 18th birthdays than during any other recorded 1 year period. We know also that Tom had his next greatest growth between his 16th and 17th birthdays. Look at the graph of h and decide when Tom had his third greatest yearly growth. Give the ages between which this growth happened and then use the table to find how much Tom grew.

2. Look at the graph of h and decide when Tom had his least yearly growth. Give the ages between which this growth happened and then use the table to find how much Tom grew.

3. (a) From the graph find at which birthday Tom's height first became greater than twice his height at birth. When did his height first become greater than three times his height at birth?

   (b) Tom's final height was how many times his height at birth? Name this number in three different ways: by a fraction, by a mixed numeral, and by a decimal numeral (rounded to the nearest hundredth).

4. (a) Make a complete table showing Tom's growth function g. Corresponding to the input 1 give the growth from age 0 to age 1. We have started the table for you. Don't forget: outputs corresponding to inputs from 3 to 12 are shown on page 85.

Table: Growth function g

Age	1	2	3	...
Growth since preceding birthday	$4\frac{1}{4}$	?	$2\frac{1}{2}$	

   (b) Why is 0 an acceptable input for the function h, but not an acceptable input for the function g?

   (c) The largest input for the function h is 21. Is it also an acceptable input for our function g?

5. (a) Draw the graph of Tom's growth function g from problem 4. Choose a scale on the vertical axis so that your graph will be about 3 inches high.

(b) Join consecutive points, belonging to the graph of g, by segments. Do these segments belong to the graph of the growth function?

(c) Compare the graphs of functions h and g for the inputs (ages) from 1 to 6. Notice that the graph of function h is increasing for these inputs while the graph of function g is decreasing. In terms of Tom's growth pattern, what does this mean?

(d) Compare the graphs of functions h and g for the inputs (ages) from 14 to 18. Notice that, for these inputs, the graphs of both functions are increasing. In terms of Tom's growth pattern, what does this mean?

6. What was Tom's average rate of growth during his growing period from birth to the age of 18 years?

Exercises 2-5c

The Evans family consists of Mr. and Mrs. Evans and three children. When the children were young Mr. and Mrs. Evans decided to start a savings fund to pay for the college education of their children. The correspondence below shows how much money was in the fund at the end of each year.

The number of years the Evans have had a savings fund	The number of dollars in the savings fund
0	$ 0
1	1,020
2	1,680
3	3,312
4	2,906
5	4,850
6	6,784
7	8,120
8	6,215
9	7,028
10	9,560
11	8,420
12	10,164

1. This correspondence between number of years and number of dollars represents a function.

   (a) How do we know that the numbers in the left-hand column are the inputs for this function?

   (b) What name can we give to the numbers in the right-hand column?

2. The graph of this function consists of a set of 13 points. With the exception of the point whose coordinates are (0,0),

   (a) in which quadrant of a coordinate plane can you draw the graph of this function?

   (b) What can be done to make the graph of this function easier to read?

3. Draw a graph of this function: Let points on the horizontal axis represent the years and points on the vertical axis represent the savings. (A convenient scale on the vertical axis is 1 unit of

measure = $500.) Join consecutive points of this graph by segments.

4. The segments, joining consecutive points of this graph, seem to have 2 different kinds of steepness.

Some segments are steep in the sense that they rise from left to right. For example,

Some segments are steep in the sense that they fall from left to right. For example,

Now, let's find meanings to assign to these segments, in terms of the Evans family and their savings fund. It seems reasonable to state that a rising segment should mean an increase in savings. What meaning should we give to a falling segment?

5. From the graph find the year in which there was the greatest increase in savings. How does the graph show this?

6. From the graph determine the year in which there was the sharpest drop in savings. How does the graph show this?

7. From the graph determine the longest period of years during which there were only increases in savings. How does the graph show this?

8. During the fourth year of savings the Evans family bought a new house. Explain how this information "fits" with the segment joining the points whose coordinates are (3, 3312) and (4, 2906).

9. During the tenth year of savings Mr. Evans received a big increase in salary. Explain how this information "fits" with the segment joining the points whose coordinates are (9, 7028) and (10, 9560).

10. (a) How much money did the Evans family save over a period of 12 years?

(b) For the entire period what was the average change in savings per year?

2-6. The Identity Function

Certainly one of the simplest of all functions is the identity function:

$$I : x \to x.$$

If we imagine this function as a machine operation, then whatever number we use as input is returned to us as output. This happens, for example, when we put a coin into a machine for cold drinks, and the machine has run out of cold drinks; the coin is returned to us. In Figure 11a, the identity function machine outputs $\bar{}17$ for the input $\bar{}17$.

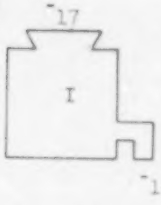


Figure 11a

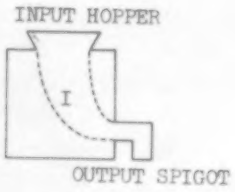


Figure 11b

It certainly ought to be easy to build such a machine. The inner workings would consist entirely of a tube connecting the input hopper and the output spigot, as shown in Figure 11b.

Graphing this function is, of course, quite simple. Take a particular input, say 3, and its corresponding output, also 3, and plot the point corresponding to (3,3). Similarly, the points corresponding to the number pairs (0,0), (1,1), (5,5), and so on, belong to this graph.

In connection with the identity function, let's examine the set of four prints: O with coordinates (0,0), B with coordinates (3,0), P with coordinates (3,3), and A with coordinates (0,3). It is clear that these points are the vertices of a square. The diagonal of this square (see Figure 12) from O(0,0) to P(3,3) divides this square into two triangles having the same shape and size, $\triangle$ AOP and $\triangle$ BOP.

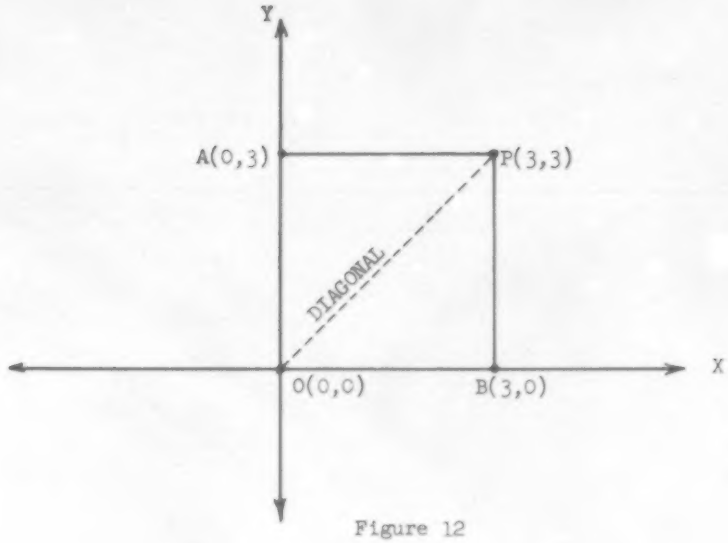


Figure 12

To understand this, think of cutting a square out of paper, as in Figure 13a, and folding the square over the diagonal $\overline{OP}$, as suggested in Figure 13b.

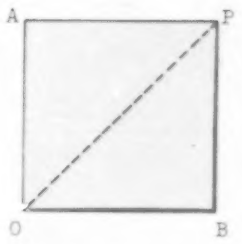


Figure 13a

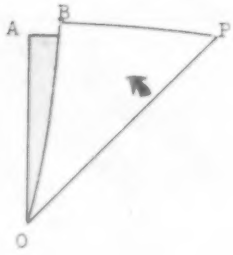


Figure 13b

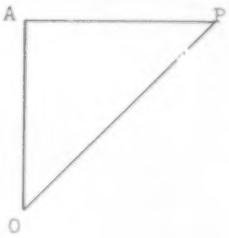


Figure 13c

After this folding, the points A and B will coincide. See Figure 13c. So will the angles: ∠AOP and ∠BOP. (You have seen this done many times in folding square napkins.) Thus the angles, ∠AOP and ∠BOP, have the same measure so that in Figure 13a the ray $\overrightarrow{OP}$ bisects the angle BOA formed by rays on the coordinate axes. The same reasoning holds for any points with the two coordinates equal such as (2,2), (4.5,4.5), (8,8), and so on.

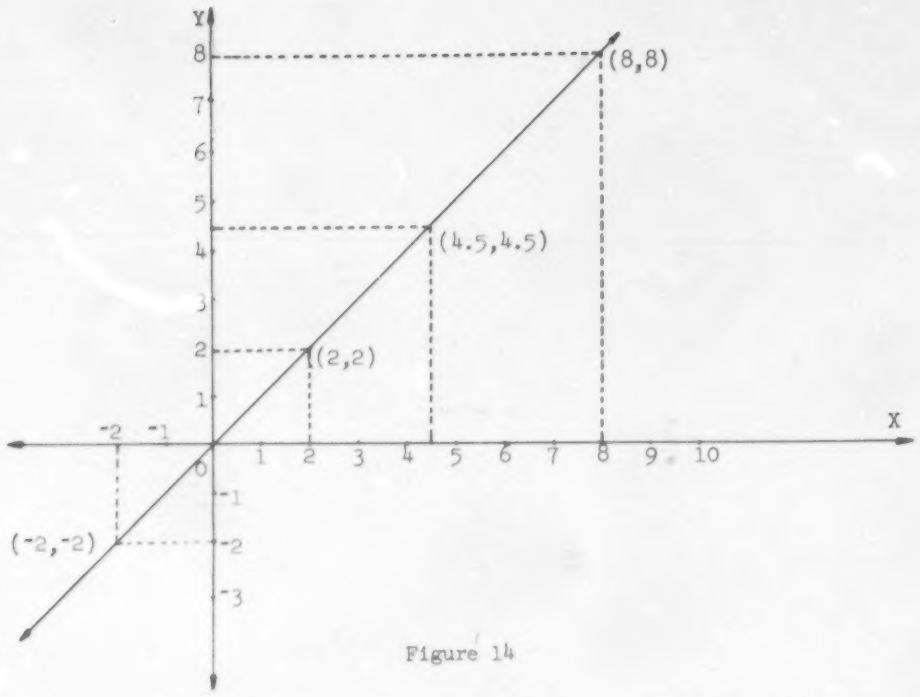


Figure 14

So all these points lie on the ray bisecting the first quadrant. If we extend this ray into the third quadrant, thus having an entire line, we see that all points on this line have both coordinates the same. Thus the graph of the identity function is a line through the diagonal of the square in Figure 12. The graph of this simple, but important, function is drawn for you in Figure 14. Study it carefully.

Exercises 2-6

1. On one coordinate plane, draw the graphs of the following functions.

   (a) $I : x \to x$

   (b) $h : x \to x + 2,\ x \geq 0$

   (c) $k : x \to x + \frac{1}{2},\ x \geq 0$

   (d) $\ell : x \to x + 5,\ x \geq 0$

2. We know that the graph of the identity function I can be described as a line.

   (a) How can we describe the graphs of functions h, k, and ℓ?

(b) How are the graphs of h, k, and ℓ related to the graph of I?

(c) Are functions I, h, k, and ℓ increasing functions? Explain.

2-7. New Kinds of Functions

Example 1:

Mr. Gray works for the National Metals Corporation. He is often called upon to work overtime and has agreed to do so provided he is paid \$4 per hour and is to get a full hour's pay for any part of an hour that he works. For example, he is to be paid for 11 hours when he works $10\frac{1}{4}$ hours. This correspondence between number of hours worked and pay received is a function. The input is the number of hours worked and the output is the corresponding pay; and corresponding to each input there is exactly one output.

If we try to describe this function by using the arrow notation we run into difficulties. We may get a clearer understanding of the function by drawing a graph. When we consider the set of inputs and the set of outputs we note that they do not follow a familiar pattern. For example,

For all inputs, n,

if $0 < n \leq 1$, then the output is 4;

if $1 < n \leq 2$, then the output is 8;

if $2 < n \leq 3$, then the output is 12;

if $3 < n \leq 4$, then the output is 16;

if $4 < n \leq 5$, then the output is 20, and so on.

Let us draw a graph of these inputs and outputs, plotting the inputs on the horizontal axis and the outputs on the vertical axis. Note that we are using different scales on the horizontal and vertical axes.

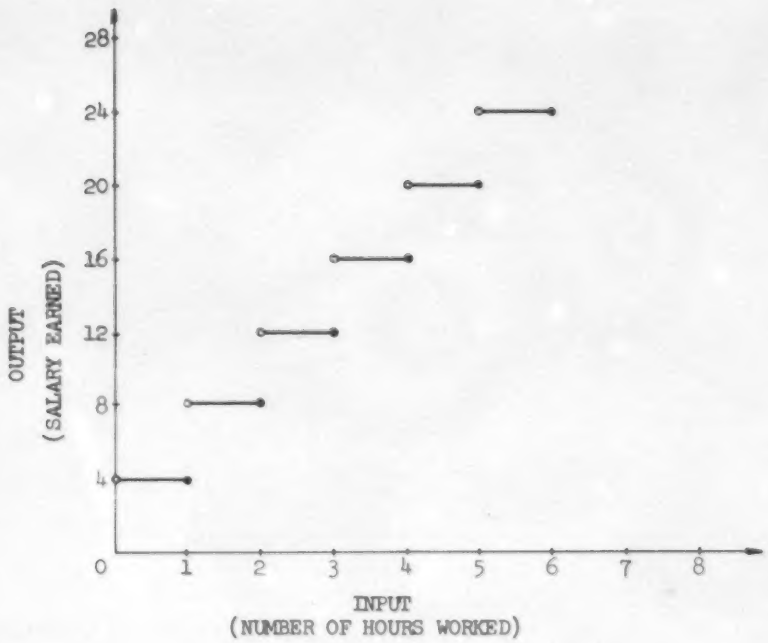


Figure 15

The tiny circle at the left of each horizontal segment shows us that this endpoint does not belong to the graph.

We see that the graph of this function is made up completely of a number of horizontal pieces. A function whose graph has this property is called a step function.

Example 2:

In the manufacture of certain machine parts at the National Metals Corporation it is necessary to cut 1-foot lengths of steel out of large sheets of steel of standard width. In this process, a leftover piece of steel with length less than 1 foot is considered waste. Thus, for a sheet of steel of standard width that is 6 feet 5 inches in length, six machine parts may be obtained. The extra 5 inches are waste. This correspondence between lengths of steel and the numbers of machine parts obtained is a function. The set of inputs consists of the various steel sheet lengths used and the set of outputs consists of the numbers of machine parts obtained.

Again, a graph may help us to get a clearer understanding of the function. The inputs and outputs do not follow a familiar pattern. For example,

For all inputs n,

if $0 \leq n < 1$, then the output is 0;

if $1 \leq n < 2$, then the output is 1;

if $2 \leq n < 3$, then the output is 2;

if $3 \leq n < 4$, then the output is 3;

if $4 \leq n < 5$, then the output is 4; and so on.

Let us draw a graph of this function, plotting the inputs on the horizontal axis and the outputs on the vertical axis.

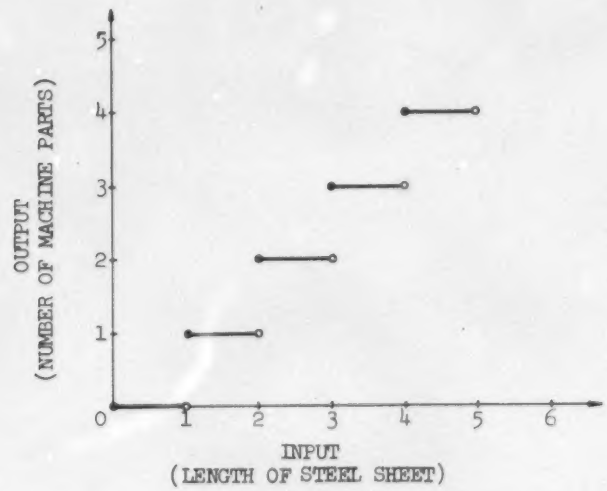


Figure 16

The tiny circle at the right of each horizontal segment shows us that this endpoint does not belong to the graph.

Exercises 2-7a

(Class Discussion)

1. We know, from Example 1, that the correspondence between number of hours worked and salary earned is a new kind of function called a step function.

   (a) When is a function called a step function?

   (b) In Example 2, is the correspondence between length of sheet steel and number of machine parts a step function?

   (c) In Figure 16, why are different scales used on the horizontal and vertical axes?

There are functions whose graphs are obtained by repeatedly sliding an initial portion of the graph to the right always by the same amount. Such a function is called a periodic function. The number of units by which we must slide the first portion to get the second and slide the second to get the third, and so forth, is called the period of the function. The function whose graph is drawn in Figure 17 is a periodic function whose period is 1.

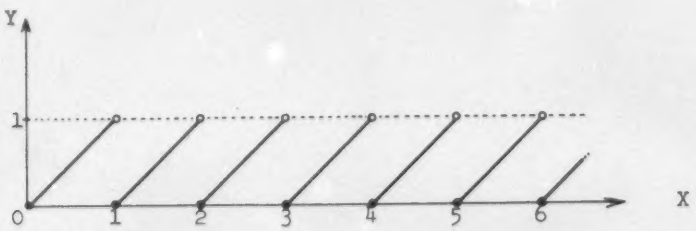


Figure 17

Note that over the interval from 0 to 1 the graph of this function is the same as that of the identity function. The rest of the graph can be thought of as being obtained by successively shifting this portion, between 0 and 1, one unit to the right. If a function has a period of 1, and x is an input, the output corresponding to an input of $x + 1$ is the same as the output associated with an input of x.

Of course, there are periodic functions with periods other than 1. For example, Figure 18 shows the graph of a periodic function with a period of 2.

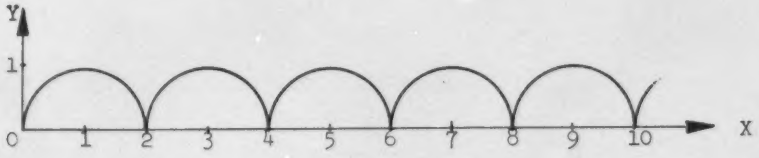


Figure 18

We can see that any one section of the graph must be shifted two units to the right to fit the next section of the graph.

The function whose graph is shown in Figure 19 is a periodic function whose period is 3. Any one section of the graph must be shifted three units to the right to fit the next section of the graph.

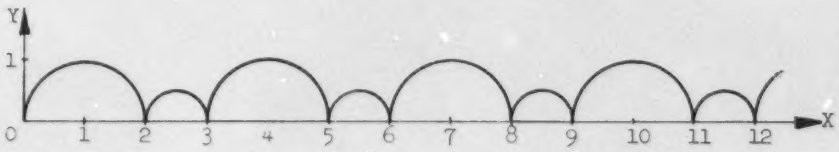


Figure 19

If a function has period 2, and x is an input, the output will be the same as the output for an input of $x + 2$. Likewise, if a function has period 3, and x is an input, the output will be the same as the output for an input of $x + 3$.

Exercises 2-7b

1. The Ace Car Rental Service rents a certain type of car for $7 per day with the agreement that use for any part of a day will carry a charge for the full day.

   (a) This statement involves a function. Below are 3 columns of correspondences belonging to this function. Study each column separately.

$\frac{1}{4} \longrightarrow 7$ $\quad 1\frac{1}{4} \longrightarrow 14$ $\quad 2\frac{1}{24} \longrightarrow 21$

$\frac{1}{2} \longrightarrow 7$ $\quad 1\frac{1}{2} \longrightarrow 14$ $\quad 2\frac{3}{24} \longrightarrow 21$

$1 \longrightarrow 7$ $\quad 1\frac{3}{4} \longrightarrow 14$ $\quad 3 \longrightarrow 21$

How do we know that this function is a step function?

(b) Draw the graph of this step function.

2. An electrical repair service charges \$10 for the first hour and \$6 per hour, or fraction of an hour, after the first hour.

(a) Write 5 inputs and their corresponding outputs for this function.

(b) Draw the graph of this function.

3. The cost of sending a letter by air mail in the United States is 10 cents per ounce or fractional part of an ounce. Draw a graph of this function.

4. Draw a graph of the function

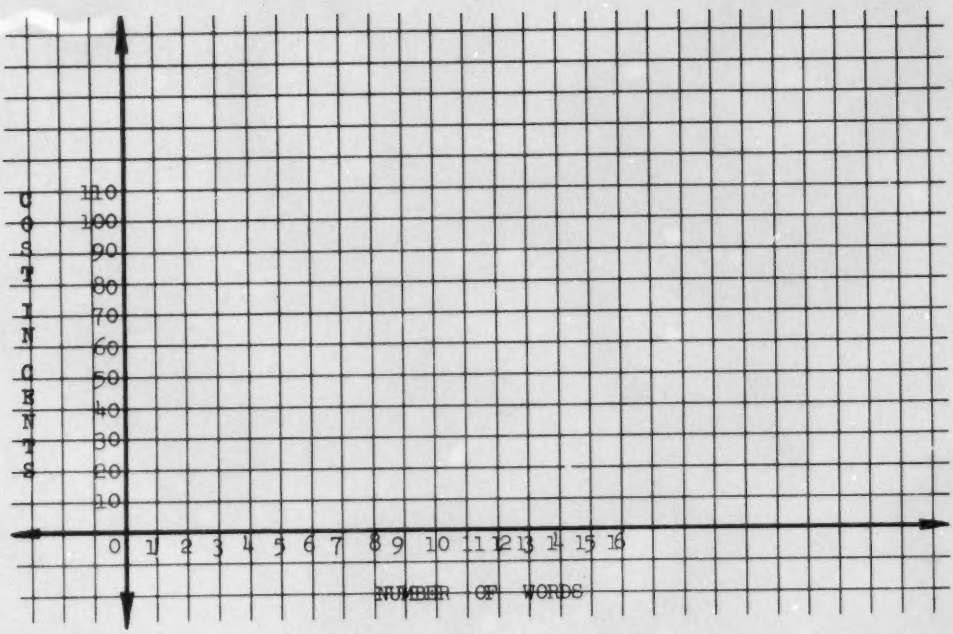
$$f : n \to \begin{cases} 1 & \text{if } 0 < n \leq 3, \\ 2 & \text{if } n > 3. \end{cases}$$

5. Consider the function described by the following statement:

> The cost of sending a telegram between two given points is 50 cents for the first 10 words (or less) and 5 cents for each word over 10.

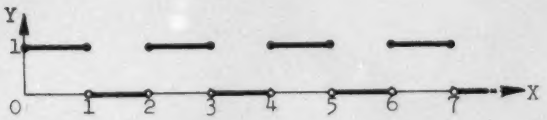
(a) Represent this function using arrow notation.

(b) Draw the graph of the function on a coordinate system like the one on the following page.

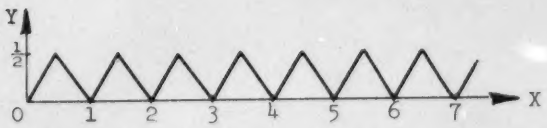


6. What are the periods of the functions whose graphs are shown below?

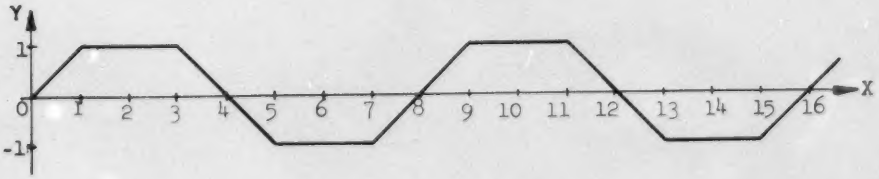
(a)



(b)



(c)



7. (a) The function whose graph is shown in 6(a) has two <u>different</u> outputs. What are the outputs for this function?

(b) The function whose graph is shown in 6(b) has the output $\frac{1}{2}$ corresponding to an indefinite number of different inputs. If x represents the smallest input whose output is $\frac{1}{2}$, then what counting number must be <u>repeatedly added to</u> <u>x</u> to find all other inputs which have the image $\frac{1}{2}$?

(c) Look at the graph shown in 6(c), and decide what output will correspond to <u>any</u> input between, and including, 21 and 23.

2-8. <u>Summary</u>

<u>Section 2-1.</u>

An introduction to functions was given in this section by way of a discussion on automobile travel using the familiar words, distance, speed, and time. For a given speed, we found that distance is a function of time; for example,

if the car travels 50 miles per hour, the distance, $d = 50t$;
if the car travels 45 miles per hour, the distance, $d = 45t$.

The average speed (in miles per hour) is the total distance traveled divided by the number of hours spent in travel.

<u>Section 2-2.</u>

If a heavy stone is dropped from a cliff or a building, it travels farther during the second one-second interval than during the first one, farther during the third one-second interval than during the second one, etc. In this way, a new rule, modeled on an actual physical problem, was introduced. Again, we have the distance as a function of time. The distance which a falling body, or falling object, drops can be approximated by

$$d = 16t^2.$$

A new symbol, $\approx$, meaning "approximately equal to", was introduced, and the above rule, or formula, is sometimes written

$$d \approx 16t^2.$$

Section 2-3.

A function is a special kind of correspondence such that corresponding to each input, there is exactly one output.

It is sometimes helpful to imagine a function as a kind of machine.

Arrow notation is a convenient way of representing a function in brief form.

The arrow notation $f : n \rightarrow 2n$ can be read as "the function f which associates with each input n, the output $2n$".

If a correspondence is a function, the reverse correspondence may or may not be a function.

Section 2-4.

The text has introduced you to 5 ways, other than arrow notation, for representing a function:

(1) a statement,

(2) a table,

(3) a diagram,

(4) a formula,

(5) a graph.

"To draw the graph of a function" means to find a collection of points in the coordinate plane that will accurately picture the special kind of association or correspondence which we call a function.

When drawing the graph of a function, any unit of measure may be selected for the X-axis, and any unit of measure for the Y-axis.

Section 2-5.

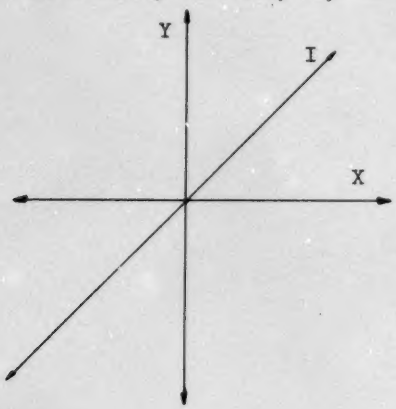
The graph of a function very often shows the correspondence more clearly than a table.

Certain properties of a function can be easily discovered by examining its graph.

Section 2-6.

A simple, but important, function is the identity function, $I : x \to x$.

The graph of the identity function, I, is a line.



Section 2-7.

A step function is a function whose graph is made up entirely of a number of horizontal pieces.

A periodic function is a function whose graph can be drawn by finding that part of the graph which constantly repeats itself, and then sliding this part to the right always by the same amount.